

Homework 3-4

Starred problems due on Friday, 9 November

3.1 Let ABC be a triangle and let f be an isometry. Prove that the points C and D lie on the same side with respect to the line AB if and only if the points $f(C)$ and $f(D)$ lie on the same side with respect to the line $f(A)f(B)$.

3.2 Let $f, g \in \text{Isom}(\mathbb{E}^2)$. Show that $g(x) \in \text{Fix}_{gfg^{-1}} \Leftrightarrow x \in \text{Fix}_f$ for all $x \in \mathbb{E}^2$.

3.3 (*) Show that the map

$$f(\mathbf{x}) = A\mathbf{x}, \quad A \in GL_2(\mathbb{R}),$$

is an isometry if and only if $A \in O_2(\mathbb{R})$ (i.e. $A \in GL_2(\mathbb{R})$, $A^T A = I$).

3.4 Let $f : z \mapsto 2z$, $z \in \mathbb{C}$. Let G be a group of transformations of \mathbb{E}^2 generated by f .

- (a) Does G act discretely on \mathbb{C} ? Justify your answer.
- (b) Show that G acts discretely on $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.
- (c) Find a fundamental domain for the action $G : \mathbb{C}^*$.

3.5 (*) Let P be a regular hexagon on \mathbb{E}^2 .

- (a) Find a group H acting on \mathbb{E}^2 discretely and such that P is a fundamental domain for the action $H : \mathbb{E}^2$. (Describe the group in terms of its generators).
- (b) Let G be a group generated by reflections with respect to the sides of P . Show that G is discrete.

The following three subquestions are a bit more involved and are **not mandatory** for submission. You are very welcome to write down and show me your solution/sketches/ideas but also don't worry if you are not really sure how to do that.

- (c) Find a fundamental domain for G .
- (d) Is H a subgroup of G ? If yes, find its index $[G : H]$.
- (e) Describe the orbit space of the action $H : \mathbb{E}^2$.

Hint: if you were not too creative in part (a) you would probably get some space we have already seen in the course.

4.1 Let $G : \mathbb{E}^2$ be a cyclic group generated by a translation T . Let X be an orbit space of $G : \mathbb{E}^2$.

- (a) Show that X is an infinitely long cylinder which admits a Euclidean metric (i.e. each point on X has a neighbourhood isometric to a domain in \mathbb{E}^2).
- (b) Find a closed geodesic on X ;
- (c) Find an open geodesic on X .

4.2 Let X be a torus obtained by identification of opposite sides of the Euclidean square.

- (a) Are there closed geodesics on X ?
- (b) Are there open ones?

4.3 (*) Given ruler and compass and a circle C on the plane, construct the centre of the circle. You can use without proofs and further descriptions the construction of perpendicular bisector for a given segment.

4.4 (*) Does there exist a map of a domain on the sphere onto a domain on the Euclidean plane that takes the segments of spherical lines into segments of Euclidean lines?

References:

1. For material of Lecture 5 look at
 - G. Jones, *Algebra and Geometry*, Lecture notes (Section 1).
2. You will find (almost) all material of Lecture 6 in
 - N. Peyerimhoff, *Geometry III/IV*, Lecture notes (Section 1.6).
3. Material of Lecture 7 may be found in Kiselev's *Geometry / Book II. Stereometry*
 - A. P. Kiselev, *Geometry / Book II. Stereometry*.
4. Starting from Lecture 8 we will follow
 - V. V. Prasolov, *Non-Euclidean Geometry*
(our Lecture 8 is a first half of Prasolov's Lecture 1).