Outline

1 Euclidean geometry

1.1 Isometry group of Euclidean plane, $Isom(\mathbb{E}^2)$.

A distance on a space X is a function $d: X \times X \to \mathbb{R}$, $(A, B) \mapsto d(A, B)$ for $A, B \in X$ satisfying

1. $d(A, B) \geq 0$ $\quad (d(A, B) = 0 \Leftrightarrow A = B)$;
2. $d(A, B) = d(B, A)$;
3. $d(A, C) \leq d(A, B) + d(B, C)$ (triangle inequality).

We will use two models of Euclidean plane:
a Cartesian plane: $\{(x, y) \mid x, y \in \mathbb{R}\}$ with the distance $d(A_1, A_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$;
a Gaussian plane: $\{z \mid z \in \mathbb{C}\}$, with the distance $d(u, v) = |u - v|$.

Definition 1.1. A Euclidean isometry is a distance-preserving transformation of $\mathbb{E}^2$,
i.e. a map $f : \mathbb{E}^2 \to \mathbb{E}^2$ satisfying $d(f(A), f(B)) = d(A, B)$.

Thm 1.2. (a) Every isometry of $\mathbb{E}^2$ is a one-to-one map.
(b) A composition of any two isometries is an isometry.
(c) Isometries of $\mathbb{E}^2$ form a group (denoted $Isom(\mathbb{E}^2)$)
with composition as a group operation.

Example 1.3: Translation, rotation, reflection in a line, glide reflection are isometries.

Definition 1.4. Let ABC be a triangle labelled clock-wise. An isometry f is orientation-preserving
if the triangle $f(A)f(B)f(C)$ is also labelled clock-wise.
Otherwise, f is orientation-reversing.

Proposition 1.5. (correctness of Definition 1.12)
Definition 1.4 does not depend on the choice of the triangle ABC.

Example 1.6. Translation and rotation are orientation-preserving,
reflection and glide reflection are orientation-reversing.

Remark 1.7. Composition of two orientation-preserving isometries is orientation-preserving;
composition of an or.-preserving isometry and an or.-reversing one is or.-reversing;
composition of two orientation-reversing isometries is orientation-preserving.

Proposition 1.8. Orientation-preserving isometries form a subgroup (denoted $Isom^+(\mathbb{E}^2)$) of $Isom(\mathbb{E}^2)$.

Theorem 1.9. Let ABC and $A'B'C'$ be two congruent triangles.
Then there exists a unique isometry sending A to A', B to B' and C to C'.

Corollary 1.10. Every isometry of $\mathbb{E}^2$ is a composition of at most 3 reflections.
(In particular, the group $Isom(\mathbb{E}^2)$ is generated by reflections).

Remark: the way to write an isometry as a composition of reflections is not unique.

Example 1.11: rotation and translation as a composition of two reflections.
Glide reflection as a composition of a reflection in some line and
a translation along the same line (a composition of 3 reflections).

Theorem 1.12. (Classification of isometries of $\mathbb{E}^2$) Every non-trivial isometry of $\mathbb{E}^2$ is of one of the following four types: reflection, rotation, translation, glide reflection.

Definition 1.13. Let f $Isom(\mathbb{E}^2)$. Then the set of fixed points of f is $Fix_f = \{x \in \mathbb{E}^2 \mid f(x) = x\}$.

Example 1.14: Fixed points of Id, reflection, rotation, translation and glide reflection are
$\mathbb{E}^2$, the line, a point, $\emptyset$, $\emptyset$ respectively.

Remark. Fixed points together with the property of preserving/reversing the orientation uniquely determine the type of the isometry.

Proposition 1.15. Let $f, g \in Isom(\mathbb{E}^2)$. (a) $Fix_{gfg^{-1}} = gFix_f$;
(b) gfg^{-1} is an isometry of the same type as f.

1.2 Isometries and orthogonal transformations

Proposition 1.15a. A linear map $f : \mathbf{x} \to A\mathbf{x}$, $A \in GL(2,\mathbb{R})$ is an isometry if and only if $A \in O(2)$, orthogonal subgroup of $GL(2,\mathbb{R})$ (i.e. iff $A^T A = I$, where A^T is A transposed).

Proposition 1.16. (a) Every isometry f of $\mathbb{E}^2$ may be written as $f(\mathbf{x}) = A\mathbf{x} + \mathbf{t}$.
(b) The linear part A does not depend on the choice of the origin.

Example 1.17. Orthogonal matrices for a reflection (in the vertical axis) and for a rotation.

Proposition 1.18. Let $f(x) = A\mathbf{x} + \mathbf{t}$ be an isometry.
f is orientation-preserving if $detA = 1$ and orientation-reversing if $detA = -1$.

Exercise 1.19. (a) Show that any two reflections are conjugate in $Isom(\mathbb{E}^2)$.
(b) This is not the case for rotations, translations and glide reflections
(there are additional parameters in that cases).

Proposition 1.20. Geodesics on $\mathbb{E}^2$ are straight lines.

Problems class 1: a. Example of using reflections to study compositions of isometries (write everything as a composition of reflections, make you choice so that some of them cancel!).
b. Example of using reflection to find a shortest way from a point A to a river and then to a point B on the same bank.
c. Ruler and compass constructions: perpendicular bisector, perpendicular from a point to a line, midpoint of a segment, angle bisector, inscribed and circumscribed circles for a triangle.

1.3 Discrete groups of isometries acting on $\mathbb{E}^2$

Definition 1.21. A group <u>acts</u> on the set X (denoted $G : X$) if
$\forall g \in G\ \exists f_g$, a bijection $X \to X$, s.t. $f_{gh}(x) = (f_g \circ f_h)(x), \forall x \in X, \forall g, h \in G$.

Example 1.22. Action of $\mathbb{Z}$ on $\mathbb{E}^2$ (generated by one translation);
$Isom(\mathbb{E}^2)$ act on points of $\mathbb{E}^2$, lines in $\mathbb{E}^2$, circles in $\mathbb{E}^2$, pentagons in $\mathbb{E}^2$.

Definition 1.23. An action $G : X$ is <u>transitive</u> if $\forall x_1, x_2 \in X\ \exists g \in G : f_g(x_1) = x_2$.

Example. $Isom(E^2)$ acts transitively on points in $\mathbb{E}^2$ and flags in $\mathbb{E}^2$
(a <u>flag</u> is a triple (p, r, H^+) where p is a point, r is a ray starting from p,
and H^+ is a choice of a half-plane with respect to the line containing the ray r);
$Isom(E^2)$ does not act transitively on the circles or triangles.

Definition 1.24. Let $G : X$ be an action.
An <u>orbit</u> of $x_0 \in X$ under the action $G : X$ is the set $orb(x_0) := \bigcup\limits_{g \in G} gx_0$.

Example 1.25. orbits of $O(2) : \mathbb{E}^2$ (circles and one point);
orbits of $\mathbb{Z} \times Z : \mathbb{E}^2$ acting by vertical and horizontal translations (shifts of the integer lattice).

Definition 1.26. An action $G : X$ is <u>discrete</u> if none of its orbits possesses accumulation points,
i.e. given an orbit $orb(x_0)$, for every $x \in X$ there exists a disc D_x centred at x
s.t. the intersection $orb(x_0) \cap D_x$ contains at most finitely many points.

Example. (a) The action $\mathbb{Z} \times Z : \mathbb{E}^2$ is discrete;
(b) the action of $\mathbb{Z} : \mathbb{E}^1$ by multiplication is not discrete.
(c) Given an isosceles right angled triangle, one can generate a group G by reflections in its three sides. Then $G : \mathbb{E}^2$ is a discrete action.

Definition 1.27. An open connected set $F \subset X$ is a <u>fundamental domain</u> for an action $G : X$ if the sets $gF, g \in G$ satisfy the following conditions:

1) $X = \bigcup\limits_{g \in G} \overline{gF}$ (where $\overline{U}$ denotes the closure of U in X);
2) $\forall g \in G$, $g \neq e$, $F \cap gF = \emptyset$;
3) There are only finitely many $g \in G$ s.t. $\overline{F} \cap \overline{gF} \neq \emptyset$.

Definition 1.28.

An <u>orbit-space</u> X/G for the discrete action $G : X$
is a set of orbits with a distance function $d_{X/G} = \min\limits_{\hat{x} \in orb(x),\ \hat{y} \in orb(y)} \{d_x(\hat{x}, \hat{y})\}$.

Example 1.29. $\mathbb{Z} : \mathbb{E}^1$ acts by translations, $\mathbb{E}^1/Z$ is a circle.
$\mathbb{Z}^2 : \mathbb{E}^2$ (generated by two non-collinear translations), $\mathbb{E}^2/\mathbb{Z}^2$ is a torus.

1.4 3-dimensional Euclidean geometry

Model: Cartesian space (x_1, x_2, x_3), $x_i \in \mathbb{R}$, with distance function

$$d(x,y) = (\sum_{i=1}^{3}(x_i - y_i)^2)^{1/2} = \sqrt{(x-y, x-y)}.$$

Properties: 1. For every plane α there exists a point $A \in \alpha$ and a point $B \notin \alpha$;
2. If two distinct planes α and β have a common point A then they intersect by a line containing A.
3. Given two distinct lines l_1 and l_2 having a common point, there exists a unique plane containing both l_1 and l_2.

Proposition 1.30. For every triple of non-collinear points there exists a unique plane through these points.

Definition 1.31. A distance between a point A and a plane α is $d(A,\alpha) := \min\limits_{X\in\alpha}(d(A,X))$.

Proposition 1.32. $AX_0 = d(A,\alpha)$, $X_0 \in \alpha$ iff $AX_0 \perp l$ for every $l \in \alpha$, $X_0 \in l$.

Corollary. A point $X_0 \in \alpha$ closest to $A \notin \alpha$ is unique.

Definition 1.33. (a) The point $X_0 \in \alpha$ s.t. $d(A,\alpha) = AX_0$ is called an orthogonal projection of A to α. Notation: $X_0 = proj_\alpha(A)$.

(b) Let α be a plane, AB be a line, $B \in \alpha$, and $C = proj_\alpha(A)$.
The angle between the line AB and the plane α is $\angle(AB,\alpha) = \angle ABC$,
Equivalently, $\angle(AB,\alpha) = \min\limits_{X\in\alpha}(\angle ABX)$.

Remark. Definition 1.31 (b) and Remark 1.32 imply that if $AC \perp \alpha$ then $AC \perp l$ for all $l \in \alpha$, $C \in l$.

Definition 1.34. The angle $\angle(\alpha,\beta)$ between two intersecting planes $\alpha$and β is the angle between their normals.
Equivalently, if $B \in \beta$, $A = proj_\alpha(B)$, $C = proj_l(A)$ where $l = \alpha \cap \beta$, then $\angle(\alpha,\beta) = \angle BCA$.

Exercise: 1. Check the equivalences.
2. Let γ be a plane through BCA. Check that $\gamma \perp \alpha$, $\gamma \perp \beta$.

Proposition 1.35. Given two intersecting lines b and c in a plane α, $A = b \cap c$, and a line a, $A \in a$, if $a \perp b$ and $a \perp c$ then $a \perp \alpha$ (i.e. $a \perp l$ for every $l \in \alpha$).

Theorem 1.36. (Theorem of three perpendiculars). Let α be a plane, $l \in \alpha$ be a line and $B \notin \alpha$, $A \in \alpha$ and $C \in l$ be three points. If $BA \perp \alpha$ and $AC \perp l$ then $BC \perp l$.

2 Spherical geometry

Geometry of the surface of the sphere.

Model of the sphere S^2 in $\mathbb{R}^3$: (sphere of radius $R = 1$ centred at $O = (0,0,0)$)

$$S^2 = \{\ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$$

2.1 Metric on S^2

Definition 2.1. A great circle on S^2 is the intersection of S^2 with a plane passing though O.

Remark. Given two distinct non-geometrically opposed points $A, B \in S^2$, there exists a unique great circle through A and B.

Definition 2.2. A distance $d(A,B)$ between the points $A, B \in S^2$ is
πR, if A is diametrically opposed to B, and
the length of the shorter arc of the great circle through A and B, otherwise.
Equivalently, $d(A,B) := \angle AOB \cdot R$ (with $R = 1$ for the case of unit sphere).

Theorem 2.4. The distance $d(A,B)$ turns S^2 into a metric space, i.e. the following three properties hold:
M1. $d(A,B) \geq 0$ $(d(A,B) = 0 \Leftrightarrow A = B)$;
M2. $d(A,B) = d(B,A)$;
M3. $d(A,C) \leq d(A,B) + d(B,C)$ (triangle inequality).

Remark. Need to prove only the triangle inequality, i.e. $\angle AOC \leq \angle AOB + \angle BOC$.

2.2 Geodesics on S^2

Defin. A curve γ in a metric space X is a geodesic if γ is locally the shortest path between its points.
More precisely, $\gamma(t) : (0, 1) \to X$ is geodesic
if $\forall t_0 \in (0, 1)$ $\exists \varepsilon :$ $l(\gamma(t)|_{t_0-\varepsilon}^{t_0+\varepsilon}) = d(\gamma(t_0 - \varepsilon), \gamma(t_0 + \varepsilon))$.

Theorem 2.5. Geodesics on S^2 are great circles.

Definition 2.6. A geodesic $\gamma : (-\infty, \infty) \to X$ (where X is a metric space) is called
closed if $\exists T \in \mathbb{R} :$ $\gamma(t) = \gamma(t + T)$ $\forall t \in (-\infty, \infty)$;
and open, otherwise.

Example. In $\mathbb{E}^2$, all geodesics are open, each segment is a shortest path.
In S^2, all geodesics are closed, one of the two segments of $\gamma \setminus \{A, B\}$ is the shortest path (another one is not shortest if A and B are not antipodal).

From now on, by lines in S^2 we mean great circles.

Proposition 2.7. Every line on S^2 intersects every other line in exactly two antipodal points.

Definition 2.8. By the angle between two lines we mean an angle between the corresponding planes:
if $l_i = \alpha_i \cap S^2$, $i = 1, 2$ then $\angle(l_1, l_2) := \angle(\alpha_1, \alpha_2)$.
Equivalently, $\angle(l_1, l_2)$ is the angle between the lines $\hat{l}_1$ and $\hat{l}_2$, $\hat{l}_i \in \mathbb{R}^3$,
where $\hat{l}_i$ is tangent to the great circle l_i at $l_1 \cap l_2$ as to a circle in R^3.

Proposition 2.9. For every line l and a point $A \in l$ in this line
there exists a unique line l' orthogonal to l and passing through A.

Proposition 2.10. For every line l and a point $A \notin l$ in this line, s.t. $d(A, l) \neq \pi/2$
there exists a unique line l' orthogonal to l and passing through A.

Remark. Writing $d(A, l) \neq \pi/2$ we mean the spherical distance on the sphere of radius $R = 1$.

Definition 2.11. A triangle on S^2 is a union of three points and
a triple of the shortest paths between them.

2.3 Polar correspondence

Definition 2.12. A pole to a line $l = S^2 \cap \Pi_l$ is the pair of endpoints of the diameter DD'
orthogonal to Π_l, i.e. $Pol(l) = \{D, D'\}$.
A polar to a pair of antipodal points D, D' is the great circle $l = S^2 \cap \Pi_l$,
s.t. Π_l is orthogonal to DD', i.e. $Pol(D) = Pol(D') = l$.

Property 2.13. If a line l contains a point A then the line $Pol(A)$ contains both points of $Pol(l)$.

Definition 2.14. A triangle $A'B'C'$ is polar to ABC ($A'B'C' = Pol(ABC)$) if
$A' = Pol(BC)$ and $\angle AOA' \leq \pi/2$, and similar conditions hold for B' and C'.

Theorem 2.15. (Bipolar Theorem)
(a) If $A'B'C' = Pol(ABC)$ then $ABC = Pol(A'B'C')$.
(b) If $A'B'C' = Pol(ABC)$ and $\triangle ABC$ has angles α, β, γ and side lengths a, b, c, then $\triangle A'B'C'$ has angles $\pi - a, \pi - b, \pi - c$ and side lengths $\pi - \alpha, \pi - \beta, \pi - \gamma$.

2.4 Congruence of spherical triangles

Theorem 2.16. SAS, ASA, and SSS hold for spherical triangles.

Theorem 2.17. AAA holds for spherical triangles.

2.5 Sine and cosine rules for the sphere

Theorem 2.18. (Sine rule) $\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma}$.

Remark. If a, b, c are small than $a \approx \sin a$ and the spherical sine rule transforms into Euclidean one.

Corollary. (Thales Theorem) The base angles of the isosceles triangle are equal.

Theorem 2.19. (Cosine rule) $\cos c = \cos a \cos b + \sin a \sin b \cos \gamma$.

Remark. If a, b, c are small than $\cos a \approx 1 - a^2/2$
and the spherical cosine rule transforms into Euclidean one.

Theorem 2.20. (Second cosine rule) $\cos \gamma = -\cos \alpha \cos \beta + \sin \alpha \sin \beta \cos c$.

Remark. (a) If a, b, c are small than $\cos a \approx 1$ and the second cosine rule transforms to $\alpha+\beta+\gamma = \pi$.
(b) For a right-angles triangle with $\gamma = \pi/2$,
sine rule: $\sin b = \sin c \cdot \sin\beta$,
cosine rule: $\cos c = \cos a \cos b$ (Pythagorean Theorem).

2.6 Area of a spherical triangle

Theorem 2.21. The area of a spherical triangle with angles α, β, γ equals $(\alpha+\beta+\gamma-\pi)R^2$, where R is the radius of the sphere.

Corollary 2.22. $\pi < \alpha+\beta+\gamma \le 3\pi$,
(where the equality holds only if three vertices of the triangle lie on the same line).

Corollary 2.23. $0 < a+b+c \le 2\pi$,
(where the equality holds only if three vertices of the triangle lie on the same line).

Corollary. There is no isometry from any small domain of S^2 to a domain on $\mathbb{E}^2$

2.7 More about triangles

(1) In a spherical triangle,
(a) medians, (b) altitudes, (c) perpendicular bisectors, (d) angle bisectors are concurrent.

(2) For every spherical triangle there exists a unique circumscribed and a unique inscribed circles.

2.8 Isometries of the sphere

Example 2.25. Rotation, reflection and antipodal map.

Proposition 2.26. Every non-trivial isometry of S^2 preserving two non-antipodal points A, B is a reflection (with respect to the line AB).

Proposition 2.27. Given points A, B, C, satisfying $AB = AC$, there exists a reflection r such that $r(A) = A$, $r(B) = C$, $r(C) = B$.

Example 2.28. Glide reflection, $f = r_l \circ R_{A,\varphi} = R_{A,\varphi} \circ r_l$,
where r_l is a reflection with respect to l and $R_{A,\varphi}$ is a rotation about $A = Pol(l)$.

Theorem 2.29.
1. An isometry of S^2 is uniquely determined by the images of 3 non-collinear points.
2. Isometries act transitively on points of S^2 and on flags in S^2.
3. The group $Isom(S^2)$ is generated by reflections.
4. Every isometry of S^2 is a composition of at most 3 reflections.
5. Every orientation-preserving isometry is a rotation.
6. Every orientation-reversing isometry is either a reflection or a glide reflection.

Theorem 2.30. (a) Every two reflections are conjugate in $Isom(S^2)$.
(b) Rotations by the same angle are conjugate in $Isom(S^2)$.

Remark 2.31. Fixed points of isometries on S^2 distinguish the types of isometries.

Remark 2.32. Isometries of S^2 may be described by orthogonal matriaces 3×3.
The subgroup of or.-preserving isometries is $SO(3,\mathbb{R}) = \{A \in M_3 | A^T A = I, det A = 1\}$

Theorem 2.33. No domain on S^2 is isometric to a domain on $\mathbb{E}^2$.

3 Affine geometry

We consider the same space $\mathbb{R}^2$ as in Euclidean geometry but with larger group acting on it.

3.1 Similarity group

Similarity group, $Sim(\mathbb{R}^2)$ is a group generated by all Euclidean isometries and scalar multiplications:
$(x_1, x_2) \mapsto (kx_1, kx_2)$, $k \in \mathbb{R}$.
Its elements may change size, but preserve the following properties:
angles, proportionality of all segments, parallelism, similarity of triangles.

Remark. A map which may be written as a scalar multiplication in some coordinates in $\mathbb{R}^2$ is called homothety (with positive or negative coefficient depending on the sign of k).

Example 3.1. Using similarity to prove the following statement:
"A midline in a triangle is twice shorter than the corresponding side."

3.2 Affine geometry

Affine transformations are all transformations of the form $f(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ where $A \in GL(2, \mathbb{R})$.

Proposition 3.2. Affine transformations form a group.

Example 3.3. Affine map may be a similarity but may be not.

Affine transformations do not preserve length, angles, area.

Proposition 3.4. Affine transformations preserve
(1) collinearity of points;
(2) parallelism of lines;
(3) ratios of lengths on any line;
(4) concurrency of lines;
(5) ratio of areas of triangles (so ratios of all areas).

Proposition 3.5. (1) Affine transformations act transitively on triangles in $\mathbb{R}^2$.
(2) An affine transformation is uniquely determined by images of 3 non-collinear points.

Example 3.6. Using the affine group to prove that the medians of Euclidean triangle are concurrent.

Theorem 3.7. Every bijection $f : \mathbb{R}^2 \to \mathbb{R}^2$ preserving collinearity of points, betweenness and parallelism is an affine map.

Remark. If f is a bijection $\mathbb{R}^2 \to \mathbb{R}^2$ preserving collinearity, then it preserves parallelism and betweenness.

Theorem 3.7'. (The fundamental theorem of affine geometry)
Every bijection $f : \mathbb{R}^2 \to \mathbb{R}^2$ preserving collinearity of points is an affine map.

Corollary 3.8. If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a bijection which takes circles to circles, then f is an affine map.
If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a bijection which takes ellipses to ellipses, then f is an affine map.

4 Projective geometry

4.1 Projective line, $\mathbb{RP}^1$

Points of the projective line are lines though the origin O in $\mathbb{R}^2$.

Group action: $GL(2, \mathbb{R})$ acts on $\mathbb{R}^2$ mapping a line though O to another line through O.
So, acts on $\mathbb{RP}^2$.

Homogeneous coordinates: a line though the O is determined by a pair of numbers (ξ_1, ξ_2), $(\xi_1, \xi_2) \neq (0, 0)$,
where pairs (ξ_1, ξ_2) and $(\lambda\xi_1, \lambda\xi_2)$ determine the same line, so are considered equivalent.
The ratio $(\xi_1 : \xi_2)$ determine the line and is called homogeneous coordinates of the corresponding point in $\mathbb{RP}_1$.

The $GL(2, \mathbb{R})$-action in homogeneous coordinates writes as

$$A : \ (\xi_1 : \xi_2) \mapsto (a\xi_1 + b\xi_2 : c\xi_1 + d\xi_2), \text{ where } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and is called a projective transformation.

Remark. Projective transformations are called this way since they are compositions of projections (of one line to another line from a point not lying on the union of that lines).

Lemma 4.1. Let points $A_2.B_2, C_2, D_2$ of a line l_2 correspond to the points A_1, B_1, C_1, D_1 of the line l_1 under the projection from some point $O \notin l_1 \cup l_2$. Then $\frac{C_1A_1}{C_1B_1} / \frac{D_1A_1}{D_1B_1} = \frac{C_2A_2}{C_2B_2} / \frac{D_2A_2}{D_2B_2}$.

Definition 4.2. Let A, B, C, D be four points on a line l, and let a, b, c, d be their coordinates on l.
The value $[A, B, C, D] := \frac{c-a}{c-b} / \frac{d-a}{d-b}$ is called the cross-ratio of these points.

Lemma 4.1'. Projections preserve cross-ratios of points.

Definition 4.3. The cross-ratio of four lines lying in one plane and passing through one point is the cross-ratio of the four points at which these lines intersect an arbitrary line l.

Remark. By Lemma 4.1', Definition 4.3 does not depend on the choice of the line l.

Proposition 4.4. Any composition of projections is a liner-fractional map.

Proposition 4.5. A composition of projections preserving 3 points is an identity map.

Lemma 4.6. Given $A, B, C \in l$ and $A', B', C' \in l'$, there exists a composition of projections which takes A, B, C to A', B', C'.

Theorem 4.7. (a) The following two definitions of projective transformations of $\mathbb{RP}^1$ are equivalent:
(1) Projective transformations are compositions of projections;
(2) Projective transformations are linear-fractional transformations.

(b) A projective transformation of a line is determined by images of 3 points.

4.2 Projective plane, $\mathbb{RP}^2$

Model: Points of $\mathbb{RP}^2$ are lines through the origin O in $\mathbb{R}^3$.
Lines of $\mathbb{RP}^2$ are planes through O in $\mathbb{R}^3$.
Group action: $GL(3,\mathbb{R})$ (acts on $\mathbb{R}^3$ mapping a line though O to another line through O).
Homogeneous coordinates: a line though the O is determined by a triple of numbers (ξ_1, ξ_2, ξ_3), where $(\xi_1, \xi_2, \xi_3) \neq (0,0,0)$;
triples (ξ_1, ξ_2, ξ_3) and $(\lambda\xi_1, \lambda\xi_2, \lambda\xi_3)$ determine the same line, so are considered equivalent.
Projective transformations in homogeneous coordinates:
$A: \ (\xi_1 : \xi_2, \xi_3) \mapsto (a_{11}\xi_1 + a_{12}\xi_2 + a_{13} : a_{21}\xi_1 + a_{22}\xi_2 + a_{23}\xi_3 : a_{31}\xi_1 + a_{32}\xi_2 + a_{33}\xi_3)$,
where $A = (a_{ij}) \in GL(3,\mathbb{R})$.

Remark. (1) A unique line passes through any given two points in $\mathbb{RP}^2$.
(2) Any two lines in $\mathbb{RP}^2$ intersect at a unique point.
(3) A plane through the origin in $\mathbb{R}^3$ may be written as $a_1x_1 + a_2x_2 + a_3x_3 = 0$.
This establishes **duality** between points and lines in $\mathbb{RP}^2$
(the point (a_1, a_2, a_3) is dual to the plane $a_1x_1 + a_2x_2 + a_3x_3 = 0$).

Theorem 4.8. Projective transformations of $\mathbb{RP}^2$ preserve cross-ratio of 4 collinear points.

Definition. A triangle in $\mathbb{RP}^2$ is a triple of non-collinear points.

Proposition 4.9. All triangles of $\mathbb{RP}^2$ are equivalent under projective transformations.

Definition. 4.10. A quadrilateral in $\mathbb{RP}^2$ is a set of four points, no three of which are collinear.

Proposition 4.11. For any quadrilateral in $\mathbb{RP}^2$ there exists a unique projective transformation which takes Q to a given quadrilateral Q'.

Proposition 4.12. A bijective map from $\mathbb{RP}^2$ to $\mathbb{RP}^2$ preserving projective lines is a projective map.

Corollary 4.13. A projection of a plane to another plane is a projective map.

Remark 4.14. (Conic sections).
Quadrics, i.e. the curves of second order on $\mathbb{R}^2$ (ellipse, parabola and hyperbola) may be obtained as conic sections (sections of a round cone by a plane).
All of them are equivalent under projective transformations.

4.3 Hyperbolic geometry: Klein model

Model: in interior of unit disc.
• points - points; • lines - chords • distance: $d(A,B) = \frac{1}{2}\big|ln|[A,B,X,Y]|\big|$
where X, Y are the endpoints of the chord through AB and $[A,B,X,Y]$ is the cross-ratio.

Remark: 1. Axioms of Euclidean geometry are satisfied (except for Parallel Axiom).
2. Parallel axiom is obviously not satisfied:
Given a line l and a point $A \notin l$, there are infinitely many lines l' s.t. $A \in l$ and $l \cap l' = \emptyset$.

Theorem 4.15. The function $d(A,B)$ satisfies axioms of distance, i.e.
1) $d(A,B) \geq 0$ and $d(A,B) = 0 \Leftrightarrow A = B$;
2) $d(A,B) = d(B,A)$
3) $d(A,B) + d(B,C) \geq d(A,C)$.

Isometries of Klein model

Theorem 4.16. There exists a projective transformation of the plane that
- maps a given disc to itself,
- preserves cross-ratios of collinear points;
- maps the centre of the disc to an arbitrary inner point.

Corollary 4.17. Isometries act transitively on the points of Klein model.
Isometries act transitively on the flags in Klein model.

Remark. 1. In general, angles in Klein model are not represented by Euclidean angles.
2. Angles at the centre are Euclidean angles.
3. Right angles are shown nicely in the Klein model.

Proposition 4.18. Let l and l' be two lines in the Klein model.
Let t_1 and t_2 be tangent lines to the disc at the endpoints of l.
Then $\quad l \perp l' \;\Leftrightarrow\; t_1 \cap t_2 \in l'$.

Pairs of lines in hyperbolic geometry: two lines in hyperbolic geometry are called
intersecting if they have a common point inside hyperbolic plane;
parallel if they have a common point on the boundary of hyperbolic plane;
divergent or ultra-parallel otherwise.

Proposition-Exercise. Any pair of divergent lines has a unique common perpendicular.

4.4 Some classical theorems on projective plane

Remark on projective duality:

point $A = (a_1 : a_2 : a_3)$	$\longleftrightarrow$	line $l_A: \quad a_1x_1 + a_2x_2 + a_3x_3 = 0$
$A \in l_B$	$\longleftrightarrow$	$B \in l_A$
line through A, B	$\longleftrightarrow$	point of intersection: $l_A \cap l_B$
3 collinear points	$\longleftrightarrow$	3 concurrent lines
...	$\longleftrightarrow$	...

Proposition 4.19. (On dual correspondence) The interchange of words "point" and "line" in any statement about configuration of points and lines related by incidence does not affect validity of the statement.

Theorem 4.20. (Pappus' theorem). Let a and b be lines, $A_1, A_2, A_3 \in a$, $B_1, B_2, B_3 \in b$.
Let $P_3 = B_1A_2 \cap A_1B_2$, $P_2 = B_1A_3 \cap A_1B_3$, $P_1 = B_3A_2 \cap A_3B_2$.
Then the points P_1, P_2, P_3 are collinear.

Remark 4.21. (Dual statement to Pappus' theorem)
Let A and B be points and a_1, a_2, a_3 be lines through A, b_1, b_2, b_3 be lines through B.
Let p_1 be a line through $b_2 \cap a_3$ and $a_2 \cap b_3$,
p_2 be a line through $b_1 \cap a_3$ and $a_1 \cap b_3$,
p_3 be a line through $b_2 \cap a_1$ and $a_2 \cap b_1$.
Then the lines p_1, p_2, p_3 be concurrent.
[This is actually the same statement as Pappus' theorem.]

Remark 4.22. Pappus' theorem is a special case of Pascal's Theorem:
If A, B, C, D, E, F lie on a conic then the points $AB \cap DE$, $BC \cap EF$, $CD \cap FA$ are collinear.
[Without proof.]

Theorem 4.23. (Deasargues' theorem). Suppose that the lines joining the corresponding vertices of triangles $A_1A_2A_3$ and $B_1B_2B_3$ intersect at one point S.
Then the intersection points $P_1 = A_2A_3 \cap B_2B_3$,
$P_2 = A_1A_3 \cap B_1B_3$, $P_3 = A_1A_2 \cap B_1B_2$ are collinear.

4.5 Topology and metric of $\mathbb{RP}^2$

Remark 4.24: topology of the projective plane (contains Möbius band, non-orientable, one-sided).

Remark 4.25: Metric on the projective plane: locally isometric to S^2;
not preserved by projective transformations, so, irrelevant for projective geometry.
Geometry of $\mathbb{RP}^2$ with spherical metric (and a group of isometries acting on the space) is called **elliptic geometry** and has the following **properties:**

1. A unique line passes through any two distinct points;
2. Any two lines intersect in a unique point;
3. Given a line l and a point A (not a pole of l),
   there exists a unique line l' s.t. $A \in l$ and $l' \perp l$.

4.6 Polarity on $\mathbb{RP}^2$ (Non-examinable section!)

Consider a trace of a cone $\mathbf{C} = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2\}$ on the projective plane $\mathbb{RP}^2$ - a conic.

Definition. Points $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ of $\mathbb{RP}^2$ are called polar with respect to $\mathbf{C}$ if $a_1b_1 + a_2b_2 = a_3b_3$.

Example: points of $\mathbf{C}$ are self-polar.

Definition. Given a point $A \in \mathbb{RP}^2$, the set of all points X polar A is the line $a_1x_1 + a_2x_2 - a_3x_3 = 0$, it is called the polar line of A.

How to find the polar line:

Lemma 4.26. A tangent line to $\mathbf{C}$ at a point $B = (b_1, b_2, b_3)$ is $x_1b_1 + x_2b_2 = x_3b_3$.

Proposition 4.27. Let A be a point "outside" $\mathbf{C}$,
let l_B and l_C be tangents to $\mathbf{C}$ at B and C, s.t. $A = l_B \cap l_C$.
Then BC is the line polar to A.

Proposition 4.28. If $A \in \mathbf{C}$ then the tangent l_A at A is the polar line to A.

Proposition 4.29. Let A be a point "inside" the conic $\mathbf{C}$. Let b and c be two lines through A. Let B and C be the points polar to the lines b and c. Then BC is the line polar to A with respect to $\mathbf{C}$.

Remark 4.30. 1. Polarity generalize the notion of orthogonality.

2. More generally, for a conic $\mathbf{C} = \{\mathbf{x} \in R^3 \mid \mathbf{x}^T A\mathbf{x} = 0\}$, where A is a symmetric 3×3 matrix, the point $\mathbf{a}$ is polar to the point $\mathbf{b}$ if $\mathbf{a}^T A\mathbf{b} = 0$.
3. We worked with a diagonal matrix $A = diag\{1, 1, -1\}$.
4. If we take an identity diagonal matrix $A = diag\{1, 1, 1\}$ we get an empty conic $x^2+y^2+z^2 = 0$, which gives exactly the same notion of polarity as we had on S^2.