Durham University
Anna Felikson

Geometry 8.03.2021

Questions for Problems classes

Here are some questions which will be probably discussed in the Problems Class (subject to change!)

Contents

1 Reflections on the plane, Geometric constructions

Problems Class 1 (20 October, 2020)

1. Let $R_{A,\varphi}$ and $R_{B,\psi}$ be rotations with $0 < \varphi, \psi \leq \pi/2$. Find the type of the composition $f = R_{B,\psi} \circ R_{A,\varphi}$.

   Hint: This is an example of using reflections to study compositions of isometries (write everything as a composition of reflections, make you choice so that some of them cancel!).

2. Let A and B be two given points in one half-plane with respect to a line l. How to find a shortest path, which starts at A then travels to l and returns to B? (How to find the point where this path will reach the line l?)

3. Ruler and compass constructions: perpendicular bisector, perpendicular from a point to a line, midpoint of a segment, angle bisector, inscribed and circumscribed circles for a triangle.

2 Group actions on $\mathbb{E}^2$

Problems Class 2 (3 November, 2020)

0. Let $g_1, \dots, g_n$ be isometries of $\mathbb{E}^2$. Let $G = \langle g_1, \dots, g_n \rangle$ be the group generated by $g_1, \dots, g_n$ (i.e. the minimal group containing all of $g_1, \dots, g_n$). Show that the group G acts on $\mathbb{E}^2$.

1. Let G be a group generated by two reflections on $\mathbb{E}^2$. When G is discrete?

2. Let T be a triangle with angles $\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}$. Let r_1, r_2, r_3 be the reflection with respect to the sides of T, and let G be the group generated by r_1, r_2, r_3. In the lecture we have checked that $G : \mathbb{E}^2$ discretely. Find the fundamental domain of this action.

3. Find the orbit-space for the action introduced in Question 2.

4. Let X be a regular triangle on $\mathbb{E}^2$. Let r_1 and r_2 be two distinct reflections taking X to itself. Find the fundamental domain of the action $G : X$. Find also the orbit-space.

5. Let G be a group generated by rotation through angle $\frac{2\pi}{3}$ on the plane. Find the orbit-space of the action $G : \mathbb{E}^2$. Are there closed geodesics in this orbit-space? Are there bi-infinite open geodesics?

3 Spherical Geometry

Problems Class 3 (17 November, 2020)

1. Let $G : S^2$ be an action. G acts discretely if and only if $|G| < \infty$.
2. Let $G : X$ be an action and suppose that F is its fundamental domain. Then one can show that the action $G : X$ is discrete.
3. Let g be a reflection, $h \in Isom(S^2)$. h is a reflection if and only if there exists $f \in Isom(S^2)$ such that $fgf^{-1} = h$.
4. Let S^2 be a sphere of radius 1. Show that the length of a circle or(spherical) radius r equals to $2\pi \sin r$.

   Remark: for the sphere of radius R, the length of the circle of radius r will be $2\pi R \sin(\frac{r}{R})$. When $R \to \infty$ we see that $\frac{r}{R} \to 0$ and, hence, $2\pi R \sin(\frac{r}{R}) \to 2\pi r$.
5. Let S^2 be a sphere of radius R. Let α and β be two parallel planes crossing S^2. Find the area of the part of S^2 lying between the planes α and β.
6. One can also discuss ruler and compass constructions, as in $\mathbb{E}^2$.

4 Projective geometry

Problems Class 4 (1 December, 2020)

1. Find a projective transformation f which takes

$$\begin{aligned} A &= (1:0:0) \text{ to } (0:0:1) \\ B &= (0:1:0) \text{ to } (0:1:1) \\ C &= (0:0:1) \text{ to } (1:0:1) \\ D &= (1:1:1) \text{ to } (1:1:1) \end{aligned}$$

   Find the image of $X = AD \cap BC$ under this transformation.
2. Find $[A, B, C, D]$ for the points above. (Does it exist?)
   For $E = (1 : 1 : 0)$, $F = (1 : 2 : 0)$ find $[A, B, E, F]$.
3. Check explicitly, that the transformation f from Question 1 preserves the value of $[A, B, E, F]$.
4. Let A_1, A_2, A_3, A_4 be points on a line a, let B_1, B_2, B_3, B_4 be points on a line b. Denote by p_i the line through A_i and B_i. Show that if the lines p_1, p_2, p_3, p_4 are concurrent, then the points $A_{i+1}B_i \cap A_iB_{i+1}$ $(i = 1, 2, 3)$ are collinear.
5. Formulate and prove the statement dual to the one in Question 4.

5 Möbius geometry

Problems Class 5 (26 January, 2021)

1. Find the type of Möbius transformation $f(z) = 1/z$.
2. Let f and g be inversions with respect to two intersecting circles. Show that $g \circ f = f \circ g$ if and only if $Fix_f \perp Fix_g$.
3. Let $\mathcal{C}_1, \dots, \mathcal{C}_5$ be circles all passing through the points A and B on the plane. Show that there exists a circle γ orthogonal to all of $\mathcal{C}_i$.
4. Prove Ptolemy's theorem: given a quadrilateral $ABCD$ inscribed into a circle one has

$$|AB| \cdot |CD| + |BC| \cdot |AD| = |AC| \cdot |BD|.$$

6 Poincaré disc

Problems Class 6 (9 February, 2021)

1. Let l_1 and l_2 be two divergent lines. Show that there exists a unique common perpendicular to l_1 and l_2.
2. Let $0 \le \alpha, \beta, \gamma < \pi$, $\alpha + \beta + \gamma < \pi$. Then there exists a hyperbolic triangle with angles α, β, γ.
3. Show that every triangle in $\mathbb{H}^2$ has an inscribed circle.
4. Given a hyperbolic triangle, is it true that it always has a circumscribed circle?
5. Ruler and compass constructions in hyperbolic plane:
   - midpoint of a segment;
   - perpendicular bisector;
   - angle bisector;
   - centre of a given circle;
   - tangent line to a given circle;
   - centre of inscribed circle for a triangle;
   - centre of circumscribed circle for a triangle (when exists);
   - common perpendicular for two divergent lines (in assumptions that one is given an infinitely long ruler, which allows to draw lines through two points of the absolute).

7 Some computations in hyperbolic plane

Problems Class 7 (22 February, 2021)

1. Show that area of hyperbolic disc of radius r is $4\pi\sinh^2(\frac{r}{2})$.

2. [Hyperbolic oranges] Consider hyperbolic oranges, i.e. a discs of radius r, where the inner disc of radius $\frac{9}{10}r$ is a pulp, while the outer $1/10$ is a peal. Assuming that all oranges are equally tasty, which oranges are better to buy: big or small?

3. [Hyperbolic golf] Assume you are playing golf on hyperbolic field and your aim is 300m apart from you. You made a very good shoot, by sending a ball 300m far away with only 1^o mistake. How far away from you goal will the ball land?

4. We had many statements about hyperbolic triangles (e.g. sine and cosine rules, congruence theorems, area formula) - which of them still hold for unbounded triangles, having at least one vertex at the absolute?

8 Computations in hyperboloid model

Problems Class 8 (9 March, 2021)

1. Given a right-angled triangle with sides a, b, c and corresponding angles α, β, γ, where $\gamma = \pi/2$, show that
$$\sinh a = \sinh c \sin\alpha.$$

2. Prove Pythagorean theorem $\cosh c = \cosh a \cosh b$ for a right-angled triangle with $\gamma = \pi/2$.

3. For a right-angled triangle with $\gamma = \pi/2$ prove $\tanh b = \tanh c \cos\alpha$.

4. Let $ABCD$ be a hyperbolic quadrilateral, with $\angle A = \angle B = \angle C = \pi/2$, $AB = a$ and $BC = b$. Denote $\angle D = \varphi$. Find $\cos\varphi$.

5. Find the radius of the circle inscribed into an ideal hyperbolic triangle.

9 Revision

Revision will not be based on demonstrating solutions, but still, there will be a couple of illustrating examples in each lecture, so you can try solving them in advance.

Revision 1 (26 April, 2021)

Revision will not be based on demonstrating solutions, but still, there will be a couple of examples in each lecture, so you can try them in advance.

1. Let $ABCD$ be a trapezoid on Euclidean plane. Let M and N be the midpoints of the parallel sides BC and DA respectively. Let $P = AB \cap CD$ and $Q = AC \cap BD$. Show that the points M, N, P, Q are collinear.

2. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two intersecting circles. Suppose they are not tangent. Let $\mathcal{C}_3$ be a circle orthogonal to both $\mathcal{C}_1$ and $\mathcal{C}_2$. And let $\mathcal{C}_4$ and $\mathcal{C}_5$ be circles tangent to all of $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ (touching the same segment of $\mathcal{C}_3$). Show that $\mathcal{C}_4 \cap \mathcal{C}_5 \neq \emptyset$.

Revision 2 (29 April, 2021)

1. Let γ be a circle in $\mathbb{H}^2$. Suppose that $f(\gamma) = \gamma$ for some isometry $f \in Isom(\mathbb{H}^2)$. Describe f.

2. Let $G = \langle g_1, g_2 \rangle$, where $g_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $g_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ act on $\mathbb{R}^2$ as linear transformations. Let $A = (0,0), B = (1,0), C = (0,1)$.

   (a) Is there an element $g \in G$ such that $g(\triangle ABC) = \triangle ACB$?

   (b) Is there an element $g \in G$ such that $g(\triangle ABC) = \triangle ACD$, where $D = (0,3)$?