Solutions 1-2

2.1 (*) Let $Isom^+(\mathbb{E}^2) \subset Isom(\mathbb{E}^2)$ be a group of orientation-preserving isometries of $\mathbb{E}^2$. Show that $Isom^+(\mathbb{E}^2)$ is generated by rotations.

Solution. By Corollary 1.10 every isometry of $\mathbb{E}^2$ is a composition of at most 3 reflections. Since reflection changes the orientation, the subgroup $Isom^+(\mathbb{E}^2)$ only contains compositions of 2 reflections and the identity map (composition of 0 reflections). Let $f = r_2 \circ r_1$ be a composition of reflections with respect to the lines l_1 and l_2. If l_1 is parallel to l_2 then f is a translation, otherwise f is a rotation.

So, we only need to prove that every translation is a composition of rotations. To do that we will use reflections again! Let f be a translation along the line l, then there are two lines l_1 and l_2, both orthogonal to l and such that $f = r_2 \circ r_1$ (where r_i is the reflection with respect to l_i). Let r be the reflection with respect to the line l. Then the elements $r_2 \circ r$ and $r \circ r_1$ are rotations (by π) and f may be obtained by their composition:

$$(r_2 \circ r) \circ (r \circ r_1) = r_2 \circ r_1 = f.$$

Remark. Another solution may be obtained even without using the classification theorem: one could prove that an orientation-preserving isometry is a composition of at most two rotations - in exactly the same way as we proved that a general isometry is a composition of atmost 3 reflections.

2.2 Show that a composition of a rotation and a translation is a rotation by the same angle. How to find the centre of the new rotation?

Solution. Let R be a rotation around O by an angle α and let T be a translation. We need to investigate $T \circ R$.

Let l be the line parallel to the direction of the translation T. Let l_1 be the line through O perpendicular to l and let l_2 be the line through O such that $R = r_1 \circ r_2$, where r_i is reflection with respect to l_i (clearly, l_2 makes the angle $\alpha/2$ with l_1). Then the translation T may be seen as a composition of r_1 and some other reflection r_3 (with respect to the line l_3 parallel to l_2 and orthogonal to l): $T = r_3 \circ r_1$. So, we have

$$T \circ R = (r_3 \circ r_1) \circ (r_1 \circ r_2) = r_3 \circ r_2.$$

As l_3 is parallel to l_1, they make the same angle with l_2, so the rotation $r_3 \circ r_2$ is exactly by the same angle as $r_1 \circ r_2$. The centre of the rotation is the intersection of the lines l_2 and l_3.

2.3 A *glide reflection* is a composition of a reflection with respect to a line and a translation along the same line. Show that every composition of 3 different reflections in $\mathbb{E}^2$ is a glide reflection. What if some of the three reflections conincide?

Solution. Let $f = r_3 \circ r_2 \circ r_1$ be a composition of reflections with respect to the line l_1, l_2, l_3.

Notice that if $l_1 || l_2$ and $l_2 \perp l_3$ then f is a glide reflection by definition (as $r_2 \circ r_1$ is a translation along the line parallel to l_3). Similarly, we get a glide reflection if $l_2 || l_3$ and $l_2 \perp l_1$. We will use this in the reasoning below.

We will consider several possibilities:

1. If all three lines are intersecting in one point O, then f is a reflection. Indeed, in this case $r_2 \circ r_1$ is a rotation, so it may be represented as $r_3 \circ r_1'$ for some reflection with respect to a line l_1' forming the same angle with l_3 as l_1 is forming with l_2. Hence, in this case

$$f = r_3 \circ r_2 \circ r_1 = r_3 \circ r_3 \circ r_1' = r_1.$$

2. Now, suppose that $O = l_1 \cap l_2$ and $O \notin l_3$. Consider a pair of lines l_1', l_2' through O forming the same angle as l_1 and such that $l_2' \perp l_3$. Then $r_2 \circ r_1 = r_2' \circ r_1'$. Now, let $M = l_2' \cap l_3$. Consider a pair of mutually orthogonal lines l_2'' and l_3'' through M such that $l_2'' || l_1'$. Then $r_3 \circ r_2' = r_3'' \circ r_2''$ and we have
$$f = r_3 \circ r_2 \circ r_1 = r_3 \circ r_2' \circ r_1' = r_3'' \circ r_2'' \circ r_1'.$$
Notice that $l_2'' || r_1'$ and $l_2'' \perp l_3''$, which implies that f is a glide reflection.
3. The case when the lines l_2 and l_3 do intersect and their intersection point does not lie on l_1 is considered similarly to the case 2..
4. If all three lines are parallel to each other, then the translation $r_2 \circ r_1$ may be written as $r_3 \circ r_1'$ for some line $l_1' || l_1$. Hence,
$$f = r_3 \circ r_2 \circ r_1 = r_3 \circ r_3 \circ r_1' = r_1'$$
and f is a reflection.

2.4 (*) List all finite order elements of the group $Isom(\mathbb{E}^2)$.

Solution. Each isometry of $\mathbb{E}^2$ is a composition of at most 3 reflections.

- 0 reflections: identity is of finite order 1.
- 1 reflection: reflection is of order 2.
- 2 reflections: either translation or rotation. The former is of infinite order as it acts as $z + a$ on the complex plane - and $z + ka \neq z$ for $a \neq 0$. The rotation by angle α is of finte order if and only if $k\alpha = n \cdot 2\pi$. Hence, a rotation by angle α is of finite order is and only if $\alpha = a\pi$ where $a \in \mathbb{Q}$.
- 3 reflections: by Problem 4, it is a glide reflection, isometry of infinite order (to see that the glide reflection $f = T_t \circ r_l$ is of infinite order consider the restriction of f to the line l).

So, isometry of the Euclidean plane is of finite order if it is a reflection or a rotation by $a\pi$, $a \in \mathbb{Q}$ (we don't need to mention identity separately as identity is a rotation by 0).

2.5 Let t_a be a translation by the vector a and let $R_{\alpha,z}$ be a rotation by angle α around $z \in \mathbb{C}$. What can you say about the isometry $f = R_{\alpha,z} \circ t_a \circ R_{-\alpha,z}$?

Solution. First, as f is conjugate to t_a it is a translation. To find the vector of the translation, we represent both $R_{\alpha,z}$ and t_a as a composition of two reflections.

Let l_1 be a line through z orthogonal to the vector a, let l_2 be the line parallel to l_1 and lying on the distance $|a|/2$, so that $r_2 \circ r_1 = t_a$ (as usually, r_i is a reflection with respect to l_i). Let l_3 be a line through z such that $R_{\alpha,z} = r_3 \circ r_1$ (it makes angle $\alpha/2$ with l_1). Then
$$f = (r_3 \circ r_1) \circ (r_2 \circ r_1) \circ (r_1 \circ r_3) = r_3 \circ r_1 \circ r_2 \circ r_3 = r_3 \circ (r_1 \circ r_2) \circ r_3.$$
So, f is conjugate to t_{-a} by r_3. It is easy to check now that f translates by the vector $b = r_3(-a)$.

2.6 Give an example of an isometry $f : \mathbb{E}^2 \to \mathbb{E}^2$ and a set $A \subset \mathbb{E}^2$ for which

(a) $f(A) \subset A$, $f(A) \neq A$;
(b) A is a bounded set, $f(A) \subset A$, $f(A) \neq A$.

Solution. (a) For example, one could choose A to be a half-plane $x > 0$ and f to be a translation by $(1, 0)$.

(b) One possibility to construct the example is the following.
Let R be a rotation around $(0, 0)$ by some angle $\alpha\pi$, where $\alpha \in \mathbb{R}$ is any irrational number, say $\sqrt{2}$. We will define A to be a countable set of points on the unit circle:
$$a_1 = (1, 0) \quad \text{and} \quad a_{i+1} = R(a_i).$$
Then it is clear that A is bounded and that $f(A) \subset A$. Irrationality of α implies that $a_1 \notin f(A)$.

2.7 (*) Let $x = (x_1, x_2)$ be a point in $\mathbb{E}^2$ and $a = (a_1, a_2)$ be a vector. Consider the line given by the equation $\langle x, a\rangle = 0$, i.e. the set of points $\{(x_1, x_2) \mid a_1x_1 + a_2x_2 = 0\}$.
Show that the transformation

$$f : x \mapsto x - 2\frac{\langle x, a\rangle}{\langle a, a\rangle}a$$

(a) is an isometry;

(b) preserves the line $\langle x, a\rangle = 0$ pointwise;

(c) is a reflection with respect the the line $\langle x, a\rangle = 0$.

(d) What is the geometric meaning of $\frac{\langle x,a\rangle}{\langle a,a\rangle}a$?
(It should help you see that f is the reflection without any computations).

Solution. (a) We need to check that given two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$, the distance between x and y is the same as the distance between $f(x)$ and $f(y)$. In other words, we need to prove $\langle x - y, x - y\rangle = \langle f(x) - f(y), f(x) - f(y)\rangle$. This is a straightforward computation:

$$f(x) - f(y) = (x - 2\frac{\langle x, a\rangle}{\langle a, a\rangle}a) - (y - 2\frac{\langle y, a\rangle}{\langle a, a\rangle}a) = (x - y) - 2\frac{\langle x - y, a\rangle}{\langle a, a\rangle}a,$$

so,

$$\langle f(x)-f(y), f(x)-f(y)\rangle = \langle x-y, x-y\rangle - 4\frac{\langle x - y, a\rangle}{\langle a, a\rangle}\langle a, x-y\rangle + 4\frac{\langle x - y, a\rangle^2}{\langle a, a\rangle^2}\langle a, a\rangle = \langle x-y, x-y\rangle.$$

(b) If $\langle x, a\rangle = 0$ then $f(x) = x + 0 = x$.

(c) An isometry of $\mathbb{E}^2$ preserving a line pointwise is either identity or reflection. It is clear that f is not an identity (it moves non-trivially every point not lying in the line $\langle x, a\rangle = 0$), so, it is a reflection.

(d) The vector $x - \frac{\langle x,a\rangle}{\langle a,a\rangle}a$ is orthogonal to a (check by taking the scalar product!), so $v = \frac{\langle x,a\rangle}{\langle a,a\rangle}a$ is exactly the component of x orthogonal to the line $\langle x, a\rangle = 0$ (the absolute value would clearly work correctly if both x and a where of length 1; now, by linearity this value is proportional to the length of x and does not depend on the length of a a $\langle a, a\rangle = ||a||^2$. So, $x - v$ is the orthogonal projection to the line and $x - 2v$ is the reflection image of x.

2.8 (Mirror on the wall)
Assume you are 2m tall and looking at the wall mirror from 1m away. How long the mirror should be so that you could see both your toes and your head? How the answer depend on your hight? on the distance to the mirror?

Solution. See “Mirror on the wall” entry on cut-the-knot (you will find both the solution and the applet to play with).