Solutions 7-8

7.1. (a) For an affine transformation $f = A\mathbf{v} + \mathbf{b}$ in affine 2-dimensional space, find f^{-1}.
(b) Find an affine transformation $f(\mathbf{v}) = A\mathbf{v} + \mathbf{b}$ which maps the points $(0, 0)$, $(1, 0)$, $(0, 1)$ to the points $(0, 1)$, $(2, 4)$, $(4, 4)$ respectively.
(c) Find an affine transformation $g(\mathbf{v}) = C\mathbf{v} + \mathbf{d}$ which maps the points $(4, 7)$, $(9, 6)$, $(-2, 8)$ to the points $(0, 0)$, $(1, 0)$, $(0, 1)$ respectively.
(d) Use (b) and (c) to find an affine transformation $h(\mathbf{v}) = E\mathbf{v} + \mathbf{q}$ which maps the points $(4, 7)$, $(9, 6)$, $(-2, 8)$ to the points $(0, 1)$, $(2, 4)$, $(4, 4)$ respectively.

Solution. The question (d) may be solved by a direct computation (finding 6 variables from 6 linear eqations). The sequance of subquestions (a)-(d) presents the easier and more reliable way to do that.

(a) f is a linear transformation followed by a translation, so, f^{-1} is a translation (by $-\mathbf{b}$) followed by the inverse linear transformation, $f^{-1}(\mathbf{v}) = A^{-1}(\mathbf{v} - \mathbf{b})$.

(b) Looking at the image of $(0, 0)$ we find $\mathbf{b} = (0, 1)$. Then looking at the image of $(1, 0)$ we find the first column of A, and from the image of $(0, 1)$ we find the second column of A, we get

$$f(\mathbf{v}) = \begin{pmatrix} 2 & 4 \\ 3 & 3 \end{pmatrix} \mathbf{v} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(c) Similarly to (b) we first find the inverse transformation g^{-1}
(takes the points $(0, 0)$, $(1, 0)$, $(0, 1)$ to the points $(4, 7)$, $(9, 1)$, $(3, 8)$):

$$g^{-1}(\mathbf{v}) = \begin{pmatrix} 5 & -6 \\ -1 & 1 \end{pmatrix} \mathbf{v} + \begin{pmatrix} 4 \\ 7 \end{pmatrix}.$$

Then, using the result of (a) and the equation $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ we get

$$g(\mathbf{v}) = \begin{pmatrix} -1 & -6 \\ -1 & -5 \end{pmatrix} \left(\mathbf{v} - \begin{pmatrix} 4 \\ 7 \end{pmatrix} \right) = \begin{pmatrix} -1 & -6 \\ -1 & -5 \end{pmatrix} \mathbf{v} + \begin{pmatrix} 46 \\ 39 \end{pmatrix}.$$

(d) We find

$$h(\mathbf{v}) = f \circ g(\mathbf{v}) = \begin{pmatrix} 2 & 4 \\ 3 & 3 \end{pmatrix} \left(\begin{pmatrix} -1 & -6 \\ -1 & -5 \end{pmatrix} \mathbf{v} + \begin{pmatrix} 46 \\ 39 \end{pmatrix} \right) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -6 & -32 \\ -6 & -33 \end{pmatrix} \mathbf{v} + \begin{pmatrix} 248 \\ 256 \end{pmatrix}.$$

7.2. (*) Through each vertex of a triangle, two lines dividing the opposite side into three equal parts are drawn. Let P be the hexagon bounded by these six lines. Prove that the diagonals joining the opposite vertices of P are concurrent.

Solution. Consider an affine transformation f which takes the given triangle ABC to a regular triangle $A'B'C'$. Notice that as affine transformations preserve collinearity and the ratios of lengths on every line, the six lines in ABC will map to the lines deviding the sides of $A'B'C'$ in proportion 1:2 and 2:1. So, the image $f(P)$ is symmetric with respect to every median on $A'B'C'$. In particular, the diagonals of $f(P)$ lie on the medians of $A'B'C'$. As the medians of $A'B'C'$ are concurrent, the diagonals of $f(P)$ also intersect in one point (since $f(P)$ preserves concurrency). As f (being an affine transformation) preserves concurrency, we conclude that the diagonals of P are also concurrent.

7.3. Prove that an arbitrary convex pentagon $ABCDE$ with sides parallel to its diagonals (i.e. such that $AB||CE, BC||DA$, etc) can be affinely transformed into a regular pentagon.

Solution. Let $A'B'C'D'E'$ be a regular pentagon. Let f be an affine map which takes the points A, B, C to $A'B'C'$. As affine maps preserve the parallelism, $f(D)$ lies on the line parallel to $B'C'$ through A', i.e., $f(D)$ lies on the line $A'D'$. Similarly, $f(E)$ lies on $C'E'$. Furthemore, the side $f(DE)$ is parallel to $A'C' = f(AC)$. For each line l parallel to $A'C'$ denote $D_l = l \cap A'D'$, $E_l = l \cap C'E'$.

Let us look at the line l sliding parallely to $A'C'$ and prove that there is a unique position of this line such that $A'E_l$ is parallel to $B'D_l$. The existance side is easy: it is just the line $D'E'$. side of the regular pentagon. If we could prove that for every other position of l the lines $A'E_l$ and $B'D_l$ do intersect, we know that f takes $ABCDE$ to the regular pentagon $A'B'C'D'E'$ (as f should also preserve parallelism of AE and BD).

To show that the position of l such that $A'E_l$ is parallel to $B'D_l$ is unique, let us move the line l parallel to $A'C'$ starting from the line through $Y = A'D' \cap C'E'$ and moving away from AC. As we move, the line $A'E_l$ rotates about A' in one direction and the line $B'D_l$ rotates about B' in the other direction, and it is easy to see that $A'E_l$ does intersect $B'D_l$ when l is closer to Y than $D'E'$ is, as well as $A'E_l$ does intersect $B'D_l$ (on the other side with respect to l) when l lies further from Y than $D'E'$ does.

7.4. (*) (a) Use similarity of triangles (or any other arguments of affine geometry) to prove Theorem of Menelaus:

> *Given a triangle ABC, and a transversal line that crosses BC, AC and AB at points D, E and F respectively, with D, E, and F distinct from A, B and C, then*

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$

(Note that at least one of the sides will have to be extended to get the intersection point).

(b) The theorem was known before Manelaus. Menelaus proved the spherical version of the theorem:

$$\frac{\sin AF}{\sin FB} \cdot \frac{\sin BD}{\sin DC} \cdot \frac{\sin CE}{\sin EA} = 1.$$

Use sine low to prove the spherical version of of Theorem of Menelaus.

(a) **Solution.** Let l be the transversal line and let A', B' and C' be the orthogonal projections of the points A, B, C to l. Then from three pairs of similar triangles we get

$$\frac{AF}{BF} = \frac{AA'}{BB'} \qquad \frac{BD}{DC} = \frac{BB'}{CC'} \qquad \frac{CE}{AE} = \frac{CC'}{AA'}.$$

Taking the product of the three values we get

$$\frac{AF}{BF} \cdot \frac{BD}{DC} \cdot \frac{CE}{AE} = \frac{AA'}{BB'} \cdot \frac{BB'}{CC'} \cdot \frac{CC'}{AA'} = 1.$$

Another solution: Consider the three homothecies f, g, h with centers D, E, F that respectively send B to C, C to A, and A to B. The composition $h \circ g \circ f$ then is an element of the group of homothecy-translations that fixes B, so it is a homothecy with center B, possibly with ratio 1 (in which case it is the identity). This composition $h \circ g \circ f$ fixes the line DE if and only if F is collinear to D and E (since the first two homothecies certainly fix DE, and the third does so only if F lies on DE). Therefore, D, E, F are collinear if and only if this composition is the identity, which means that the product of the following three ratios is 1:

$$\frac{DC}{DB} = \frac{EA}{AC} = \frac{FB}{FA} = 1.$$

(b) Using sine law for triangles AFE, BFD and CED respectively we get

$$\frac{\sin AF}{\sin \angle E} = \frac{\sin AE}{\sin \angle F}, \qquad \frac{\sin BF}{\sin \angle D} = \frac{\sin BD}{\sin \angle F}, \qquad \frac{\sin CE}{\sin \angle D} = \frac{\sin CD}{\sin \angle E},$$

(here we use that $\sin AEF = \sin DEC$). Rewriting the three equations we get

$$\frac{\sin AF}{\sin AE} = \frac{\sin \angle E}{\sin \angle F}, \qquad \frac{\sin BD}{\sin BF} = \frac{\sin \angle F}{\sin \angle D}, \qquad \frac{\sin CE}{\sin CD} = \frac{\sin \angle D}{\sin \angle E}.$$

Multiplying the three equations we get the required identity.

7.5. Three pegs on a plane form an isosceles right triangle with a leg of length 3. The pegs may move to an arbitrary distance but on a line parallel to the line formed by the other two. Is it possible to eventually get the three pegs at the vertices of a right triangle with legs 2 and 4?

Solution. The admissible transformation on the triple of pegs preserves the area of the triangle. As $3 \cdot 3/2 \neq 2 \cdot 4/2$, no composition of these transformation will take the initial triangle to the right triangle with legs 2 and 4.

7.6. (a) Find a projective transformation f which takes the points $0, 1, \infty$ of the projective line to the points $3, 4, 0$ respectively.
(b) Find the image of the point 2 under the transformation f in part (a) and use it to check that f preserves the cross-ratio of the points $0, 1, 2, \infty$.

Solution. (a) We will find f as a linear-fractional transformation $f = \frac{ax+b}{cx+d}$, $a, b, c, d \in \mathbb{R}$:
Since $f(\infty) = 0$, we have $a = 0$.
Since $f(0) = 3$, we have $b/d = 3$.
Since $f(1) = 4$, we have $b/(c+d) = 4$. We can choose $b = 12$, then $d = 4$, $c = -1$.
So $f(x) = \frac{12}{-x+4}$.

(b) $f(2) = 6$, $\quad [0, 1, 2, \infty] = \frac{2}{1} / \frac{\infty}{\infty} = 2$; $\quad [f(0), f(1), f(2), f(\infty)] = [3, 4, 6, 0] = \frac{3}{2} / \frac{-3}{-4} = 2$.

7.7. Let $A, B, C, D \in \mathbb{R}^2$ be four collinear points and O be a point not on the same line. Suppose that OB is a median of AOC and OC is a median of BOD. Find the cross-ratio of the lines OA, OB, OC and OD.

Solution. The cross-ratio of four lines is the cross-ratio of their intersection points with any given line l, in particular, it coincides with $[A, B, C, D] = \frac{CA}{CB} / \frac{DA}{DB} = \frac{2}{1} / \frac{3}{2} = 4/3$.

7.8. (*) (a) Show that $[A, B, C, D] = [C, D, A, B] = [B, A, D, C] = [D, C, B, A]$.
(b) Given $[A, B, C, D] = \lambda$ find $[A, B, D, C]$ and $[A, C, B, D]$.
(c) Assuming that $[A, B, C, D] = \lambda$, find all other possible values for $[X_1, X_2, X_3, X_4]$, where (X_1, X_2, X_3, X_4) is a permutation of (A, B, C, D).

Solution. (a) The first two equalities may be checked directly (you get the same multiples in the other order). The third follows from the first two.
(b) $[A, B, D, C] = \frac{d-a}{d-b} / \frac{c-a}{c-b} = \frac{1}{[A,B,C,D]} = 1/\lambda$.

$$[A, C, B, D] = \frac{b-a}{b-c} / \frac{d-a}{d-c} = \frac{(b-a)(d-c)}{(b-c)(d-a)} = \frac{((b-c)-(a-c))((d-b)-(c-b))}{(b-c)(d-a)} =$$
$$-\frac{(a-c)(d-b)}{(b-c)(d-a)} + \frac{(b-c)((d-b)-(c-b)-(a-c))}{(b-c)(d-a)} = -\lambda + 1,$$

since $(d-b) - (c-b) - (a-c) = (d-a)$.

(c) In (b) we have seen that together with λ one gets $1/\lambda$ and $1-\lambda$ (we will denote these operations by "$-$" and "$\div$". We may apply these transformations repeatedly (but it does not make sense to apply the same transformation twice in a row - we get back to λ). So, applying these transformations one after another we get

$$\lambda \overset{-}{\longleftrightarrow} 1-\lambda \overset{\div}{\longleftrightarrow} \frac{1}{1-\lambda} \overset{-}{\longleftrightarrow} \frac{-\lambda}{1-\lambda} \overset{\div}{\longleftrightarrow} \frac{1-\lambda}{-\lambda} \overset{-}{\longleftrightarrow} \frac{1}{\lambda} \overset{\div}{\longleftrightarrow} \lambda,$$

which gives 6 different values. Taking in account that each value works for 4 permutations (see (a)) and that 24 is the number of all permumations of 4 objects, we see that the six values is the complete list.

So, all possible values are $\lambda, 1-\lambda, \frac{1}{1-\lambda}, \frac{\lambda}{1-\lambda}, \frac{1-\lambda}{\lambda}, \frac{1}{\lambda}$.

8.1. Find the projective transformation of $\mathbb{RP}^2$ sending the points $(1:0:0), (0:1:0), (1:0:1)$ and $(3:-1:2)$ to the points $(2:-1:1)$, $(0:1:1)$, $(1:0:2)$, $(0:0:1)$.

Solution. First, we will find a projective transformation f mapping the points $(1:0:0), (0:1:0), (1:0:1)$ and $(3:-1:2)$ to $(1:0:0), (0:1:0), (0:0:1), (1:1:1)$.

Let $f(v) = Av$, where $A \in GL(3, \mathbb{R})$, $v = (v, y, x)$. Since

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ d \\ g \end{pmatrix},$$

we see that $d = g = 0$. Similarly applying A to $(0,1,0)$ we see that $b = h = 0$.
Applying A to $(1,0,1)$ we get $(a+c, f, j)$ which should be proportional to $(0,0,1)$, so $c = -a$, $f = 0$.
Next, Applying A to $(3,-1,2)$ we get $(a, -e, 2j)$, which should be proportional to $(1,1,1)$. So, $a = -e = 2j$ and A may be written as

$$A = \begin{pmatrix} 2 & 0 & -2 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Next, we will find a projective transformation g mapping the points $(1:0:0), (0:1:0), (0:0:1), (1:1:1)$ to the points $(2:-1:1)$, $(0:1:1)$, $(1:0:2)$, $(0:0:1)$.

Let $g(v) = Bv$, $B \in GL(3, \mathbb{R})$. Applying B to the first theree vectors, it is easy to see that

$$B = \begin{pmatrix} 2 \cdot k & 0 \cdot l & 1 \cdot n \\ -1 \cdot k & 1. \cdot l & 0 \cdot n \\ 1 \cdot k & 1 \cdot l & 2 \cdot n \end{pmatrix}$$

for some $k, l, n \in \mathbb{R}$ (columns proportional to the vectors we want to get).
Also, applying B to the vector $(1,1,1)$ we want to get a vector proportional to $(0,0,1)$, which gives $k = l = -n/2$ and B may be written as

$$B = \begin{pmatrix} 2 & 0 & -2 \\ -1 & 1 & 0 \\ 1 & 1 & -4 \end{pmatrix}.$$

Finally, we obtain the transformation as

$$BA = \begin{pmatrix} 2 & 0 & -2 \\ -1 & 1 & 0 \\ 1 & 1 & -4 \end{pmatrix} \begin{pmatrix} 2 & 0 & -2 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 & -6 \\ -2 & -2 & 2 \\ 2 & -2 & -6 \end{pmatrix}.$$

Remark. While searching for the matrix A we have actually used that the set of initial vectors was quite similar to what we wanted to obtain (so that the matrix A is upper triangular, with many zeros). In general, one should first search for A^{-1}, matrix taking $(1:0:0), (0:1:0), (0:0:1), (1:1:1)$ to the initial vectors (this is easy, its columns are proportional to the first three vectors), and then to find A as the inverse to A^{-1}.

8.2. How many projective transformations send a quadrilateral to itself?

Solution. Let $ABCD$ be a quadrilateral. For each permutation σ of the vertices A, B, C, D there exists a unique projective transformation which sends $ABCD$ to $\sigma(A)\sigma(B)\sigma(C)\sigma(D)$. Since there are 24 permutations on the 4-element set, there are 24 projective maps taking the quadrilateral to itself.

8.3. (*) Calculate the cross-ratio of the following four points lying on the infinite line: $(1; 2; 0)$, $(2; 3; 0)$, $(3; 4; 0)$, $(4; 1; 0)$.

Solution. The cross-ratio of four collinear points in $\mathbb{RP}^2$ is the cross-ratio of the corresponding four (coplanar) lines in $\mathbb{R}^3$ (the point $(x; y; z)$ corresponds to the line in $\mathbb{R}^3$ passing through the origin and (x, y, z)).

To find the cross-ratio of the four lines (all lying in the plane $z = 0$) we find the intersections of these lines whith some line. Let l be the line given by $z = 0$, $y = 12$. Then the four lines intersect l at points $x = 6, 8, 9, 48$ respectively.

So, the cross-ratio is $\frac{3}{1} / \frac{42}{40} = 20/7$.