Feedback to Surveys

Contents

1 Michaelmas Surveys

1.1 Survey 1: Why Geometry

Thank you for numerous nice and detailed answers to the question “Why do you take Geometry this year?”. I think I should not show you any complete response, but here are some short fragments from some of your answers (I did not try to put them into any structure):

...to learn about different geometries and their properties,

...to expand my knowledge on this field and come in touch with some geometries that I haven’t learned about

I didn't have many options really!!

I expect the course to strengthen my problem solving skills and teach me to think more visually.

.. enjoy vizualising mathematics and working with diagrams etc.

...to improve my ability to write proofs

...I enjoy the geometric way of thinking and problem solving.

… looked like a really interesting part of maths

I want to learn about different ways to think about problems and how to come up with geometric solutions.

I'm interested in the subject and enjoy thinking logically.

Exciting results and pretty pictures.

...interest in non-euclidean geometries

...enjoy finding connections between different modules

I like the visual aspect

Logical process, combination of visual and abstract concepts.

I wanted to do something different from my other modules.

to study the ways in which group theory ties into Geometry.

Because it is beautiful and fun.

1.2 Survey 2: Theorems of Euclidean geometry

Feedback on Survey 2

- Q1. - There where **only 30 replies** to this one - which is not satisfactory. I think everyone in the module would benefit from going through this question in details.

  - The most "unknown" statements for you turned out to be E15, E16, E17, E19, E20, E21, E23. Here is how many votes they got:

	Statement	votes
E15	The perpendicular bisectors of the sides of a triangle are concurrent	13
E16	The angle bisectors in a triangle are concurrent	12
E17	The altitudes in a triangle are concurrent	18
E19	Let $A_1, A_2, A_3 \in OA$, $B_1, b_2, B_3 \in OB$ and $A_1B_1 \vert\vert A_2B_2 \vert\vert A_3B_3$. If $A_1A_2 = A_2A_3$ then $B_1B_2 = B_2B_3$.	18
E20	If $D \in AB$, $E \in AC$ and $DE \vert\vert BC$ then $AD : AB = AE : AC$.	9
E21	The medians of a triangle are concurrent	16
E23	In a right angled triangle ABC (with $\angle C = \pi/2$), let CD be an altitude. then $AD \cdot DB = CD^2$.	20

  - Each of the other statements was mentioned by <= 5 people.

- Q2. Thank you for all the feedback on the lectures! Thanks also for being so positive and accepting when the things are not ideal. I will try to improve on the points you mentioned.

1.3 Survey 3: Isometry group

Feedback to Survey 3

Most of your responses correctly listed the true statements.
As there were occasional mistakes, here are the correct answers:

- Orientation-preserving isometries form a group.

  Correct: we will see it in Lecture 4.

- Orientation-reversing isometries form a group.

  Incorrect:
  The set is not closed under the composition (composition of two orientation-reversing isometries is not orientation-reversing). Also, it contains no identity element.

- Isometries of E^2 taking the line $x = 0$ to the line $x = 1$ form a group.

  Incorrect: The set of such isometries contains no identity element, no inverse to some elements, for instance to $z \to z + 1$, $z \in \mathbb{C}$, and also not closed under the composition!

- Orientation-preserving isometries taking the origin to itself form a group.

  Correct: All 4 properties of groups are satisfied. The group consists of all rotations about the origin and is isomorphic to the group $G = \{z \in \mathbb{C} \mid |z| = 1\}$ of all complex numbers of absolute value 1 considered under multiplication.

- Isometries preserving the line $x = 0$ form a group.

  Correct: All 4 properties of groups are satisfied. The group consists of identity map, reflection with respect to the line, all translations along the line and glide reflections along the line.

1.4 Survey 4: Compositions of isometries

Feedback to Survey 4

In your responses, only the correct statements were listed.
But there were very few answers listing ALL correct statements.
Here are the answers:

- 1. Every isometry can be written as a composition of reflections.

  Correct: this is the result of Corollary 1.11.

- 2. Every isometry can be written as a composition of rotations.

  Incorrect: rotations preserve the orientation. So, none of the orientation-reversing isometries can be obtained this way.

- 3. Every isometry can be written as a composition of either 2 or 3 reflections.

  Correct: Corollary 1.11 states that every isometry can be written as a composition at most 3 reflections. At the same time, the identity (a composition of 0 reflections) can be written as $id = r \circ r$ - a composition of any reflection r with itself. Similarly, a reflection r (a composition of 1 reflection) can be written as $r = r \circ r \circ r$.

- 4. Every isometry can be written as a composition of at most 4 reflections.

  Correct: Directly follows from Corollary 1.11 - what you can obtain if you are allowed 3 reflections you can also obtain if you are allowed up to 4!

  [This one was missed in your answers more often than any other!]

- 5. Every orientation-reversing isometry can be written as a composition of 3 reflections.

  Correct: Similar to 3: any reflection is $r = r \circ r \circ r$.

1.5 Survey 5: More on isometries

Feedback to Survey 5

1. Let r be a reflection with respect to a line l, and let R be a rotation around a point O lying on l. What is the type of the transformation rRr?

   Solution: We know from Proposition 1.18(b) that the conjugate elements are of the same type. The conjugate element to R by r should read $r^{-1}Rr$. But $r = r^{-1}$, so rRr is conjugate to R in $Isom(\mathbb{E}^2)$. We conclude that rRr is a rotation (as R is a rotation).

   Answer: rotation.

   Statistics: all (All!!!) answers here were correct!

2 Let $f(\boldsymbol{x}) = A\boldsymbol{x}$, where A is an orthogonal 2×2 matrix. We know that this implies that $f(\boldsymbol{x})$ is an isometry. Given that $\det(A) = -1$, what is the type of isometry f?

   Solution: Since $\det(A) = -1$, we conclude that f is an orientation reversing isometry, so either reflection or glide reflection. Since $f(\boldsymbol{x}) = A\boldsymbol{x}$ it preserves the origin. Therefore, f is a reflection.

   Answer: reflection.

   Statistics: all answers named the reflection. But some also named a glide reflection, which is not right as a glide reflection does not preserve any point, while $x \to Ax$ preserves the origin.

1.6 Survey 6: Group actions

Feedback to Survey 6

The question was to determine on which of the following sets in E^2 does the group of orientation-preserving isometries act transitively?
The sets to consider where
(1) all circles; (2) circles of radius 1; (3) triangles with sides 2,3,4; (4) half-planes.

There were a lot of wrong answers this time! But everyone agreed that $Isom^+$ is not acting transitively on all circles and acts transitively on circles of radius 1. More precisely:

- **All circles:** Clearly, isometry cannot take a big circle to a small circle. So, no transitivity here.

- **Circles of radius 1:** Yes, $Isom^+$ is transitive on this set: we can take a centre of a unit circle to any given point by a translation, and then we will get the radius 1 circle with the given centre.

- **Triangles with sides 2,3,4:** After example with circle, it is clear than for transitivity one at least needs to specify the size of the Euclidean object. So, it seems everything is OK with this example. But triangles with sides 2,3,4 come with two different orientation (2,3,4 clockwise or 2,3,4 anti-clockwise). And only one of these classes can be covered by orientation-preserving isometries.

  So, no, here $Isom^+$ is **not** acting transitively.

- **Half-planes:** $Isom^+$ acts transitively on half-planes. Indeed, let l and l' be the lines and the boundary of the half-planes. We can take l to a line l'' intersecting l' by a translation, then we can rotate about $O = l'' \cap l'$ till the image of l will coincide with l' - and if the half-planes are not the same yet, we can rotate by π.

  Notice, that $Isom^+$ does not act transitively on flags in $\mathbb{E}^2$: with orientation-preserving isometries we can only guarantee that we obtain either the correct ray on l or the correct half-plane, but we cannot get both.

1.7 Survey 7: 3d geometry

Feedback to Survey 7

There were two questions:

Question 1 asked whether the following is just a reformulation of the Theorem of Three Perpendiculars:

Is it true that Theorem of Three Perpendiculars states the following: Given a point A, its orthogonal projection A' to a plane α, and the orthogonal projection A'' of A to a line $l \in \alpha$, the point A'' coincides with orthogonal projection of A' to the line l?

Most of you (but not all!) agreed that this is indeed a reformulation of the theorem
(I also think so! Please, tell me if you still think I am wrong!).
So, the answer is: **Yes.**

Question 2 asked about two orthogonal plane α and β intersecting at some line l, and about two lines $a \in \alpha$ and $b \in \beta$ passing through the same point $O \in l$. Is it true that a is always orthogonal to b?

The opinions divide half/half.

At the same time,in each of the planes α and β one can find a line a' and b' through O making a very small angle with l. Then a' and b' are obviously not orthogonal (and make a very small angle to each other!)

So, the answer is: **No.**

1.8 Survey 8: Geodesics

Feedback to Survey 8

There was just one question:

Let T be a torus obtained as an orbit space from the action of the group $G = \mathbb{Z} \times \mathbb{Z}$ on $\mathbb{E}^2$ generated by translations in two non-collinear directions. (In other words, let X be a Euclidean square with opposite sides identified). How many are there closed geodesics on T?

Statistics: About a half of those who looked at the question did not answer it... I think, the reason is that some definitions are not clear yet... Please, ask me in the Office Hours if it is the case for you! Almost all of those who answered gave the correct answer.

Solution: To think about the geodesics on the Euclidean torus, let us first look at the effect of action of the group G on $\mathbb{E}^2$. Let suppose that G is generated by translations by vectors v_1 and v_2. As we discussed in the lecture, every orbit looks as an integer lattice $\{p + k_1v_1 + k_2v_2\}$, $k_1, k_2, \in \mathbb{Z}$ (if the generating vectors are unit orthogonal vectors, then it is square lattice, otherwise, lattice made from parallelograms). When I take one point from each orbit, I can chose the representative inside the parallelogram $P = \{p + \alpha_1v_1 + \alpha_2v_2\}$, where $0 \le \alpha_i < 1$. Identifying the opposite sides of the parallelogram I get the torus T.

Now, the distance on this torus is locally measured the same way as on the plane. So, the geodesics on T are coming from straight lines on the plane: when we start a geodesic inside initial parallelogram P and then extend the line outside of P, on the quotient space we get continuation of the geodesic and it may intersect itself or turn out to be closed. To get a closed geodesic on T, we need a line on $\mathbb{E}^2$ which passes through two points of the same orbit, i.e. through two points on the integer lattice described above. Clearly, one can present infinitely many of them starting from the point $(0, 0)$ and passing through $\{p+k_1v_1+k_2v_2\}$, $k_1, k_2, \in \mathbb{Z}$.

At the same time, it does not exhaust all possible lines through the origin. Any line which is not passing through two points of a given orbit, i.e. any line which passes through the origin but not any other lattice point, will produce and open geodesic. (In terms of a square lattice, one could express this as lines with rational slopes produce closed geodesics, while the majority of lines have irrational slopes and produce open geodesic on T).

Answer: Infinitely many geodesics are closed, but not all of them.

1.9 Survey 9: Polar correspondence

Feedback to Survey 9

There were two questions:

Question 1. *Let l be a line on the sphere and let $A = Pol(l)$ be one of the two points polar to l. Let B be a point on the line l. Is it true that A lies on the line polar to B?*

Statistics: Most of you answered "Yes".

Solution: And this is right! Indeed, if $B \in Pol(A)$ then $\angle AOB = \pi/2$, and this also implies $A \in Pol(B)$.

Answer: Yes.

Question 2. *Let A and B be points on a (unit) sphere, and let $d := d(A, B)$ be the distance between them. Let a and b be lines polar to A and B respectively. Assuming that $d < \pi/2$, what is the angle between the lines a and b?*

Statistics: Everyone who attempted this question gave the same answer "d".

Solution: And this is right again! The angle between the lines a and b is the angle between the normals OA and OB to the planes containing the corresponding great circles. The later angle $\angle AOB$ is the distance between points A and B, so equals to d.

Answer: d.

1.10 Survey 10: Spherical triangles

Feedback to Survey 10

There were two questions:

Question 1. *Is there a spherical triangle with 3 obtuse angles?*

Statistics: Most of you answered "No" (and the whole experience of our Euclidean life cries for that!).

Solution: But it is wrong! One can construct a triangle with 3 obtuse angles.
For example, one can take three points on equal distance on the equator and then shift all of them slightly to the North. Then you will get a triangle with 3 angles very close to π each. Another interesting example may be obtained from a regular tetrahedron inscribed into a sphere: projecting the faces of the tetrahedron to the sphere, one obtains a tiling of the sphere by three congruent regular triangles with angles $2\pi/3$.

Answer: Yes.

Question 2. *Let ABC be a spherical triangle, let M be a midpoint of AB and N be a midpoint of BC. Is it always true that $MN < AC$?*

Statistics: Almost everyone answered "Yes" (and that would be natural, indeed...).

Solution: But in fact it does not hold for all spherical triangles.
And even $MN < 1000AC$ does not hold for all... See Example 2.24 in the lecture notes.

Answer: No.

1.11 Survey 11: Area of triangles

Feedback to Survey 11

There were two questions:

Question 1. *On a unit sphere, given a quadrilateral Q with angles $\alpha, \beta, \gamma, \delta$, the area of Q equals ...*

Statistics: Almost all answered this one correctly.

Solution: We can divide the quadrilateral into two triangles by a diagonal AC, so that the triangles will have angles $\alpha_1, \beta, \gamma_1$ and $\alpha_2, \delta, \gamma_2$, where $\alpha = \alpha_1 + \alpha_2$ and $\gamma = \gamma_1 + \gamma_2$. Then

$$S_{ABCD} = S_{ABC} + S_{ADC} = (\alpha_1 + \beta + \gamma_1 - \pi) + = (\alpha_2 + \delta + \gamma_2 - \pi) = \alpha + \beta + \gamma + \delta - 2\pi.$$

Answer: $\alpha + \beta + \gamma + \delta - 2\pi$.

Question 2. *Which of the following statements are true for polygons on the sphere? (Select all true answers):*

1. *The area of a triangle is determined by its angles.*
2. *The area of a quadrilateral is determined by its angles.*
3. *The area of a triangle is determined by its side lengths.*
4. *The area of a quadrilateral is determined by its side lengths.*

Statistics: Most of you answered (1) and (2) are correct, only a couple of you answered by the whole list (1)-(4). You will see below, that none of this is correct!

Solution:

1. Correct, as $S = \alpha + \beta + \gamma - \pi$.
2. Correct: as you said in Question 1, $S = \alpha + \beta + \gamma + \delta - 2\pi$.
3. Correct: by SSS, from the set of lengths we know the set of angles, which in its turn determine the area.
4. Wrong! Consider a quadrilateral with equal sides of the length $a < \pi/2$. We can make it very slim, almost lying along the equator - then its area is almost zero (as close to zero as we want). But we can also make fat (say turning it into a regular quadrilateral), then it will have some area $S > 0$.

Answer: 1,2,3 are correct, but 4 is wrong.

1.12 Survey 12: Isometries of S^2

Feedback to Survey 12

There was just one question:

Question 1. *We say that an isometry preserves a line if it takes the line to itself (not necessarily pointwise). Which of the following statements are correct (choose all correct statements):*

1. *An orientation-preserving isometry of a sphere fixes exactly one point.*
2. *An orientation-preserving isometry of a sphere fixes exactly one line.*
3. *There exists a non-trivial orientation-preserving isometry of the sphere which preserves all lines.*
4. *There exists an orientation-reversing isometry of the sphere which preserves all lines.*

Statistics: There was a blend of correct and incorrect answers, but almost nobody so far got it all right.

Solution:

1. Wrong. An orientation-preserving isometry is either an identity (fixed the whole S^2) or a rotation (fixes two antipodal points).
2. Wrong. An orientation-preserving isometry may be an identity, which fixes all lines.
3. Wrong. Try to prove by considering separately rotations by π and by other angles.
4. Correct. Try to find an example among the maps you have seen in the lectures!

Answer: Only the last one is correct.

1.13 Survey 13: Affine geometry

Feedback to Survey 13

There were 3 short questions:

Question 1. *Is it true that an affine map takes any circle to a circle?*

Statistics: Half of answers is Yes, half is No.

Solution: One can easily find a counterexample: for example one can use the map $x \to Ax$, where A is a diagonal matrix with 1 and 2 on the diagonal.

Answer: No.

Question 2. *A trapezoid a is a quadrilateral with a pair of parallel sides. Is it true that an affine map takes any trapezoid to a trapezoid?*

Statistics: Everyone answered Yes!

Solution: As parallelism is preserved by affine maps, the vertices in the image will also lie on two parallel lines, so the image of a trapezoid is indeed a trapezoid.

Answer: Yes.

Question 3. *Is it true that affine maps act transitively on trapezoids?*

Statistics: Almost everyone answered Yes.

Solution: Consider a triangle ABC with a midline MN parallel to AC. The quadrilateral $ACNM$ is a trapezoid. Let f be an affine map. Then $f(ACNM)$ is again a trapezoid using a midline $f(MN)$ of the triangle $f(ABC)$. In particular, if PQ is a midline of the triangle MBN parallel to MN, then the trapezoid $ACQP$ will be impossible to obtain as an image of $ACNM$ under an affine map.

Answer: No.

1.14 Survey 14: Affine maps

Feedback to Survey 14

There was just one question:

Question 1. *Which of the following statements a true? (Choose all correct statements).*

(A) An affine map is uniquely determined by images of any 3 point.

(B) An affine map is uniquely determined by images of any 4 non-collinear points.

(C) Affine maps act transitively on triples of lines through the origin.

(D) A bijection $f : R^2 \to \mathbb{R}^2$ which takes a triple of non-collinear points to a triple of non-collinear points is an affine map.

Statistics: Almost everyone got exactly half of the right answers, through some people got 3.

Solution:
A: is right by Proposition 3.5, part (2).
B: is also right (if images of 2 points determine something uniquely then fourth point cannot harm)! Unfortunately, nobody answered with both (A) and (B)...
C: It is right! when I wrote this question, I was sure I am writing an obviously wrong statement. But apparently it is right! The main ideas for the solution will be:

- It is enough to map a standard triple of lines to a given one;
- In the "given" triple l_1, l_2, l_3, chose a point $P_2 \in l_2$ and draw through P_2 the lines parallel to l_1 and l_3 to get a parallelogram with a diagonal.
- Then map a standard square to this parallelogram.

D: This is also right (the proof is very similar to the proof of Corollary 3.8).

Answer: So, all of them turned out to be correct this time...

1.15 Survey 15: Projective transformations

Feedback to Survey 15

There were two questions:

Question 1. *By Lemma 4.6 we know that any three points on a line can be mapped to a three points on another line by a composition of projections. How many projections will be sufficient for that? (Chose the smallest number which is enough in all circumstances).*

Statistics: Not many answers are there this time... And still not all of them right... (though, there are more right answers than wrong ones).

Solution: I haven answered this one in the lecture, actually... It follows from the proof of Lemma 4.6 that 2 projections are enough.

Answer: 2.

Question 2. *Which of the following ways to finish the phrase "A projective transformation of a line ...." a true? (Select all answers leading to a true statement).*

(a) is a composition of linear-fractional transformations.

(b) is a projection.

(c) is determined by images of 4 points.

(d) is determined by images of 3 points.

(e) preserve cross-ratios of points on the line.

Statistics: Various combinations of answers are given - mostly getting 3 out of 5 positions right.

Solution:

- (a) is true: a projective transformation of a line is a linear-fractional transformation, so it is a composition of linear fractional transformations.

- (b) is false: a projection preserve betweenness for points on a line, but not all projective transformation do it.
- (c) is true: by Theorem 4.7 (b) a projective transformation of a line is uniquely determined already by images of 3 points, images of 4 points contain even more information, so definitely determine a projective transformation.
- (d) is true: this is the statement of Theorem 4.7 (b).
- (e) is true: projective transformations are compositions of projections (Theorem 4.7), while projections preserve cross-ratios (Lemma 4.1').

Answer: (a),(c),(d), (e) are true, only (b) is false.

1.16 Survey 16: Projective plane

Feedback to Survey 16

There were two questions:

Question 1. *Is it true that all triple of points on the projective plane are equivalent under projective transformations?*

Statistics: Most answer where “Yes”.

Solution: However, “Yes” is wrong: collinear triple of points is not equivalent to non-collinear...

Answer: No.

Question 2. *Is it true that projective transformations of projective plane act transitively on the quadruples of collinear points?*

Statistics: Here, half of the answers were “No”.

Solution: And “No” is right here: projective transformations preserve cross-ratio of the points, so quadruples with different cross-ratios are not equivalent.

Answer: No.

1.17 Survey 17: Projective duality

Feedback to Survey 17

There was only one question:

Question 1. *Find the pairs of dual statements. (Note that the statements in the pair may be wrong or trivial!)*

One set of statements:

1. *Three points A, B, C are not lying on one line.*
2. *A point P is an intersection of three lines a, b, c.*
3. *Points A, B, C, D lie on a line q.*
4. *The diagonals e and f of a quadrilateral $ABCD$ do not intersect.*

Another set of statements:

A. Lines a, b, c, d are not concurrent.

B. Lines a, b, c, d are passing through the point Q.

C. Three lines a, b, c are not passing through one point.

D. Points A, B, C are chosen on the line p.

E. No two of the lines a, b, c, d are passing through the same point. Points E is intersection of lines a and c, point F is an intersection of lines b and d. Then E and F do not lie on the same line.

Statistics: All your answers where completely right this time!

Solution: Here are the pairs:

- - 1. Three points A, B, C are not lying on one line.
  - C. Three lines a, b, c are not passing through one point.
- - 2. A point P is an intersection of three lines a, b, c.
  - D. Points A, B, C are chosen on the line p.
- - 3. Points A, B, C, D lie on a line q.
  - B. Lines a, b, c, d are passing through the point Q.
- - 4. The diagonals e and f of a quadrilateral $ABCD$ do not intersect.
  - E. No two of the lines a, b, c, d are passing through the same point. Points E is intersection of lines a and c, point F is an intersection of lines b and d. Then E and F do not lie on the same line.

1.18 Survey 18: More on RP^2

Feedback to Survey 18

There were two questions:

Question 1. *Is it true that RP^2 contains two non-trivial closed curves which don't intersect each other?*

Statistics: "Yes", "No" and "Unanswered" were in almost equal proportion this time.

Solution: This turned out to be much harder than I have anticipated when creating the question!

I thought that the answer is "Yes" and that I have constructed the curves. But when I tried to write down an argument why these curves are non-trivial (i.e. cannot be contracted to a point, i.e. are not homo topic to a constant curve) , I have realised that (a) it is hard to show, and (b) it is not true for one of them!

Now, I think that the answer is "No", but the argument is difficult without having any tools in the hands.

One can try to argue as follow: Consider the curves up to homotopy. Let draw the curves on the disc in such a way that it intersect the boundary of the disc in the smallest possible number of points. If it does not intersect the boundary, then the curve is trivial (can be constructed inside the disc). If it intersect the boundary at a unique point and its antipode), then it is homotopic to the diameter of the disc. If it intersects the boundary at more than two points, then one can take the two adjacent intersections and get rid of them by pushing the curve first to the boundary and then on the other side, which would mean that the representative was not with minimal number of intersection with the boundary, in contradiction to our assumption. So, the only possibility is is a curve homotopic to a diameter, and it is easy to check that any two curves will intersect (as any such curve separated any point on the boundary of the disc from the antipodal point on the boundary).

Answer: So, the answer is "No", but the question was definitely too hard, not only for this "Survey" series but also for the whole module at all.

Question 2. *Is it true that in the elliptic geometry there are two points on distance π?*

Statistics: "Yes", "No" and "Unanswered" were in almost equal proportion also here.

Solution: For every point A on the sphere we can consider the line l_A polar to A. We can think of RP^2 as the half-sphere separated by l_A and containing A (with the opposite points of the boundary identified). Then it is clear that all points of RP^2 are on the distance at most $\pi/2$ from A.

Answer: No.

1.19 Survey 19: Klein model

space30pt

Feedback to Survey 19

There were two questions:

Question 1. *In the Klein model, is it true that the distance between the centre of the disc and a point on the boundary is finite?*

Statistics: Most answers were “No”.

Solution: And this directly follows from the distance formula - when one of the endpoints tends to the boundary, the cross ratio tends to infinity, and hence the distance tends to infinity.

Answer: No.

Question 2. *In the Klein disc, is it true that the rotation around the centre of the disc is an isometry?*

Statistics: Most of answers were “Yes”.

Solution: It is an isometry as it does not change Euclidean distances and hence does not change cross-ratios.

Answer: Yes.

2 Epiphany Surveys

2.1 Survey, Week 11: Möbius transformations

Feedback to Survey, Week 11

There were three questions:

Question 1. *Is it true that a Möbius transformation is determined by preimages of three points?*

Statistics: Most answers were "Yes".

Solution: And this is right! Möbius transformations form a group, so if f is a Möbius transformation, then f^{-1} is also a Möbius transformation. Given the images of three points for f, in is the same as to be given preimages of 3 points for f^{-1}, which, as we know uniquely determine the transformation f^{-1}. As f^{-1} is uniquely determined, we conclude that f is uniquely determined too.

Answer: Yes.

Question 2. *Is it true that inversion takes vertices of an acute angled triangle to vertices of an acute-angled triangle?*

Statistics: Most answers were "Yes".

Solution: Inversions preserve the angles, and it would be natural to expect that they take acute angles triangles to acute angles triangles. However, the lines are not always mapped by inversions to lines, but often they are mapped to circles. So, inversions preserve angles between the sides of potentially curvilinear triangles...

To be more precise, let γ be a circle through the centre of inversion I. Then $I(\gamma)$ is a line, which implies that the vertices of every triangle inscribed in γ are mapped to three collinear points.

Answer: No.

Question 3. *Is it true that inversion is a Möbius transformation?*

Statistics: Exactly half of answers were "Yes" and half were "No"... You will get more intuition on this from Lectures 23, 24.

Solution: So far, we considered a formula for only one inversion, i.e. $\frac{1}{\bar{z}}$ is an inversion with respect to the unit circle $|z| = 1$. One can notice that it is not a holomorphic map, in contrast to Möbius transformations.

More generally, one can notice, that even when inversion preserves the absolute values of the angles, it changes the signs of oriented angles (in the same way as reflections do), while Möbius transformations preserve angles together with signs.

Answer: No.

2.2 Survey, Week 12: Inversions

Feedback to Survey, Week 12

There were three questions:

Question 1. *Is it true that conjugating an inversion by another inversion we always obtain a reflection?*

Statistics: Almost twice more "No" answers than "Yes".

Solution: Let I_1 and I_2 be inversions with respect to circles $\mathcal{C}_1$ and $\mathcal{C}_2$. Then $I_2(\mathcal{C}_1)$ is the fixed points of $I_2 \circ I_1 \circ I_2^{-1}$. This means that is $I_2(\mathcal{C}_1)$ is a circle and not a line, then $I_2 \circ I_1 \circ I_2^{-1}$ is not a reflection. So, if $\mathcal{C}_1$ does not pass through the centre of $\mathcal{C}_2$ then $I_2 \circ I_1 \circ I_2^{-1}$ is not a reflection.

Answer: No.

Question 2. *Is it true that a composition of an inversion and a reflection preserves cross-ratios?*

Statistics: Almost all answers were "Yes".

Solution: By Remark 5.28 inversions and reflections take cross-ratios to conjugate. So, the composition of the two preserves cross-ratios.

Answer: Yes.

(See next page for Question 3.)

Question 3. *Let f be a Möbius transformation which takes any line to a line. Which of the following is true? (select all true answers).*

(1) f takes ∞ to ∞;

(2) f takes every circle to a circle;

(3) f is isometry;

(4) f^{-1} takes every line to a line.

Statistics: About half answers were completely right.

Solution:

(1) Suppose that ∞ is not preserved. Then there exists a point $z \in \mathbb{C}$ such that $f(z) = \infty$. Let l be a line not containing the point z. Then $f(l)$ cannot be a line as it does not contain ∞. The contradiction show that $f(\infty) = \infty$.

(2) We know that Möbius transformations map circles and lines to circles and lines. Since ∞ is preserved by f, a circle cannot be mapped to a line (it did not contain ∞, so its images should not). Therefore, the image of a circle is a circle.

(3) f is not necessarily an isometry: it is a transformation given by $f(z) = az + b$, where $|a|$ is not necessarily equal to 1. So, $f(z)$ can scale all distances by $|a|$ (but all distances are scaled by the same number).

(4) Since f preserves ∞, we conclude that f^{-1} preserves ∞. So preimage of a line passes through ∞, and hence is a line (as it only can be a line or a circle, since f^{-1} is a Möbius transformation.

Answer: (1), (2), (4).

2.3 Survey, Week 13: Stereographic projection

Feedback to Survey, Week 13

There were two questions:

Question 1. *Which of the following ways to finish the statement are correct? (chose all correct options).*
A preimage of a line under Stereographic projection...

1. *can be a circle;*
2. *is always a circle;*
3. *can be a great circle;*
4. *is always a great circle.*

Statistics: There was some mistake in the system this time - and I was not able to see your answers - but now I see them!
Almost everybody's answer to this question was not completely right (but I am happy to write this "almost" here!).

Solution: Stereographic projection is a restriction of inversion. So the inverse to it is also a restriction of an inversion. Inversion takes a line to a line or a circle. A preimage of stereographic projection lies on a sphere, so it cannot be a line. So, the preimage of a line under stereographic projection is always a circle.

Next, a preimage of a line always contains the North Pole of the sphere (as the line contains the point at infinity). So, it is a great circle only when it also contains the South Pole, i.e. when the line on the plane passes through the origin. So, it is not always a great circle, but can be a great circle (for each line l through the origin it is the great circle obtained as the intersection of the sphere with the plane through the origin and the line).

Answer: 1,2,3.

Question 2. Let I be an inversion in a circle $\mathcal{C}$. Let $\mathcal{C}'$ be another circle or a line. Is it true that $\mathcal{C}' = I(\mathcal{C}')$ if and only if $\mathcal{C}'$ is orthogonal to $\mathcal{C}$?

Statistics: Most answers were "Yes", but there were still several "No"...

Solution: Let $A, B \in \mathcal{C} \cap \mathcal{C}'$. Let a_1 and a_2 be two (small pieces of) arcs of $\mathcal{C}'$ starting from A, let b be a (small piece of) arc of $\mathcal{C}$ starting from A. Then $\angle(a_1, b) = \angle(I(a_1), b)$ (as inversion preserves angles), while $\angle(a_1, b) + \angle(b, a_2) = \pi$. This implies that $I(a_1)$ may coincide with a_2 only when the circles are orthogonal.

Now, suppose that $\mathcal{C} \perp \mathcal{C}'$. Then $I(\mathcal{C}')$ is a line or circle passing through A and B and orthogonal to $\mathcal{C}$. Notice that the tangent line to $\mathcal{C}$ at A and B should pass through the centre of such a circle. So, the centre is defined by the intersection of these tangent lines. So, the circle is uniquely defined by its centre and fact that it contains A and B. So, there is no other circle orthogonal to $\mathcal{C}$ and passing through A, B, except for the circle $\mathcal{C}'$. This implies that $I(\mathcal{C}') = \mathcal{C}'$.

Answer: Yes (and we have extensively used it in the lectures).

2.4 Survey, Week 14: Poincaré disc

Feedback to Survey, Week 14

There were two questions:

Question 1. *On a hyperbolic plane, a line and a circle can intersect in... (choose all possible answers)*

0. 0 points;

1. 1 points;

2. 2 points;

3. 3 points.

Statistics: Most answers were "0,1,2", but there were also some "0" and "1,2".

Solution: In the Poincaré disc model, circles are represented by Euclidean circles, and lines are represented by circles or lines. Moreover, after applying an isometry taking a point of the given line to the origin, we may assume that the line is represented by Euclidean line. These Euclidean objects (diameter and a circle) cannot have more than two intersections. At the same time, it is easy to make 0,1 and 2 examples (just draw a diameter in the disc and a small circle disjoint from it, or tangent, on intersecting at two points - the later one, say by a circle centred at the origin).

Answer: 0,1,2.

Question 2. Which of the following ways to continue the phrase lead to true statements? (choose all correct answers).

Isometries of hyperbolic plane...

(a) ... act transitively on the points of hyperbolic plane.

(b) ... act transitively on triples of points of hyperbolic plane.

(c) ... act transitively on triples of points of the absolute.

(d) ... are Möbius transformations.

Statistics: Half of participants gave correct answers. The ones who were wrong added some wrong statement to the list of the correct ones.

Solution: Isometries act transitively on the points of $\mathbb{H}^2$ by Theorem 6.9 (2).

Isometries do not act transitively on triples of points of $\mathbb{H}^2$: the cannot take a triple of points on very small distances between them to a triple of points on large distances. By this reason they are not transitive even on pairs of points...

Isometries act transitively on triples of points of absolute by Theorem 6.9 (1).

Finally, not all isometries are Möbius transformations. There are also orientation-reversing isometries, they are given by anti-Möbius transformations.

Answer: (a) and (c).

2.5 Survey, Week 15: Angle of parallelism

Feedback to Survey, Week 15

There were two questions:

Question 1.

Let ABC be a triangle with right angle C and vertex A lying on the absolute. Which of the following statements a correct? (Choose all)

(1) Angle B is the angle of parallelism for the line AC and the point A.

(2) Angle B is the angle of parallelism for the line AC and the point B.

(3) Angle C is the angle of parallelism for the line BC and the point C.

Statistics: About half of participants prefered not to answer this question. Everyone else gave the answer (2).

Solution: Angle of parallelism of a point A with respect to line l ($A \notin l$) is some angle at A. This excludes answers (1) and (3). Furthermore, it is the acute angle at A between the lines parallel to l and perpendicular to l - which is exactly what we have in (2)

Answer: (2): Angle B is the angle of parallelism for the line AC and the point B.

Question 2. Let φ be the angle of parallelism for the line CA and a point B. Let ψ be the angle of parallelism for the line CB and point A. Which of the following is true:

(a) $\varphi < \psi$;

(b) $\varphi = \psi$;

(c) $\varphi > \psi$;

(d) none of the above.

Statistics: The question was not correctly formulated - I forgot to specify that this question is in the same assumptions as Question 1, i.e that ABC is a triangle with right angle C and vertex A lying on the absolute.

Therefore, the answers a looking quite random...

Solution: The question, as it was formulated, should be answered "none of the above" - as it does not contain any information about the point A, B, C and depending on their positions we can make any of φ and ψ larger than the other one (A and B play completely symmetric roles in the statement as it was, and so, φ and ψ are completely symmetric...).

Now, if to think about the question as it was intended - in the assumptions of Question 1 - then φ is some non-zero angle, while $\psi = 0$ (angle at $A \in \partial\mathbb{H}^2$ between the line orthogonal to BC and the line parallel to BC and containing A). So, $\varphi > \psi$.

Answer: depends on how to understand the question...

What I really wanted to ask:

Let ABC be a triangle with right angle C and vertex A lying on the absolute. Let φ be the angle of parallelism for the line CA and a point B. Let ψ be the angle of parallelism for the line AB and point C. What is bigger, φ or ψ?

2.6 Survey, Week 16: Area of triangle and the hemisphere

Feedback to Survey, Week 16

There were two questions:

Question 1. *Which if the following is right (choose all):*

(a) An ideal triangle has larger area than any non-ideal triangle.

(b) An ideal quadrilateral has area larger than any non-ideal quadrilateral.

(c) An ideal triangle has area larger than any non-ideal polygon

Statistics: Almost all answers here were completely right.

Solution: (a) and (b) follows directly from the formula for area of a triangle (as ideal polygons have zero angles). For the quadrilateral here, we can divide it into two triangles and then to use the formula.

(c) is clearly wrong: we can take a "very large" quadrilateral, with vertices approaching the absolute. Then its area will be almost 2π, which is larger than the area of the ideal triangle.

Answer: (a) and (b).

Question 2. *Which of the following is true? (list all)*
In a hemisphere model ...

(a) hyperbolic lines are represented by Euclidean circles;

(b) hyperbolic lines are represented by great circles;

(c) hyperbolic lines are represented by all circles orthogonal to the boundary;

(d) hyperbolic circles are represented by all circles contained in the hemisphere.

Statistics: About half of the answers where completely right. Others mostly skipped one of the right options.

Solution: (a) is right, you consider the sections of the hemisphere by vertical planes, which gives circles.
(b) is wrong - not all of the circles you obtain are great (only the ones for which the initial chords on the Klein disc where diameters).
(c) is right - intersection of the sphere with the vertical plain is contained in the vertical plane and hence is orthogonal to the boundary.
(d) is wrong - for example, consider a circle lying in some horizontal plane.

Answer: (a) and (c).

2.7 Survey, Week 17: hyperboloid model

Feedback to Survey, Week 17

There was just one question:

Question 1. *Let $\langle x, y\rangle$ be the pseudo-scalar product. Which of the following situations are possible (select all!) ?*

(a) $\langle u, u\rangle < 0$, $\langle v, v\rangle < 0$, *and* $\langle u, v\rangle = 0$.

(b) $\langle u, u\rangle < 0$, $\langle v, v\rangle > 0$, *and* $\langle u, v\rangle = 0$.

(c) $\langle u, u\rangle > 0$, $\langle v, v\rangle > 0$, *and* $\langle u, v\rangle = 0$.

(d) $\langle u, u\rangle = 0$, $\langle v, v\rangle = 0$, *vectors* u *and* v *are not proportional, and* $\langle u, v\rangle = 0$.

Statistics: Most of the answers were correct!

Solution: In all situations we look at $Q = \frac{(\langle u,v\rangle)^2}{\langle u,u\rangle\langle v,v\rangle}$ and apply Theorems 7.3 and 7.4.

(a) When $\langle u, u\rangle < 0$, $\langle v, v\rangle < 0$, both u and v correspond to some points of $\mathbb{H}^2$, and $\cosh^2 d(u, v) = Q$. Since the distance between two different points is positive (and $\cosh d$ is positive two), we conclude that $\langle u, v\rangle \neq 0$.

(b) When $\langle u, u\rangle < 0$, $\langle v, v\rangle > 0$, u correspond to a point, while v correspond to a line. The point can lie on that line, in that case one would have $\langle u, v\rangle = 0$.

(c) When $\langle u, u\rangle > 0$, $\langle v, v\rangle > 0$, both u and v represent lines, and Q computes $\cos^2\theta$ where θ is the angle between them. So, one has $\langle u, v\rangle = 0$ when the corresponding lines are orthogonal to each other.

(d) When $\langle u, u\rangle = 0$, $\langle v, v\rangle = 0$, both u and v correspond to points at the absolute. If u is not proportional to v, these are two distinct points, hence lying on infinite distance. This implies that Q should be infinite (one can see it by considering u and v as limits of some points one the line inside $\mathbb{H}^2$ lying on the line uv). This implies that $\langle u, v\rangle \neq 0$.

Alternatively, one can argue as follows. In the remark between Theorems 7.2 and 7.3 we considered the pseudo-orthogonal compliments of the vectors with positive, zero and negative pseudo-scalar squares respectively. And we concluded that:

(a) for negative vectors such a compliment does not intersect the interior of the cone, so contains no negative vectors;

(d) for zero vectors, such a compliment is tangent to the cone at along that vector, so contains no other zero vectors;

(b,c) for positive vectors such a compliment intersects the cone and contains both positive and negative vectors.

See also Fig. 101.

Answer: (b) and (c).

2.8 Survey, Week 18: Isometries, horocycles and equidistant curves

Feedback to Survey, Week 18

There ware three questions:

Question 1. *How many different types of non-trivial isometries are on hyperbolic plane?*

Statistics: There were not many participants this time - but all gave correct answer to this question.

Solution: There are 3 types of orientation preserving isometries (elliptic, parabolic and hyperbolic) and two types of orientation-reversing ones (reflections and glide reflections). So, 5 in total.

Answer: 5.

Question 2. *Given two tangent horocycles, is it always true that their common point lies on the line through their centres?*

Statistics: again, all answers were correct.

Solution: Without loss of generality, we may assume that the line through the centres of the horocycles is $O\infty$ in the upper half-plane. The horocycle centred at ∞ is represented by a horizontal plain. The horocycle centred at 0 is represented by a circle $|z - ir| = r$ for some $r \in \mathbb{R}_+$. So, the common point is $2ir$, which lies on the line 0∞. Also, after applying

Remark. After applying an isometry $z \to z/r$ we cay map the configuration to the one with $r = 1$. We conclude that isometries act transitively on the pairs of tangent horocycles.

Answer: Yes.

Question 3.

Let l and m be two lines and e and f be equidistant curves to these lines lying on the same distance d. Which of the following statements are right (select all).

(a) If l and m are ultra-parallel then e does not intersect f.

(b) If l is parallel to m then e has a unique intersection with f.

(c) If l intersects m then e has more than one intersection with f.

(d) If l intersects m then e has exactly two points of intersection with f.

Statistics: There were some right answers and some not completely right.

Solution: (a) Wrong: if the distance d is large enough (more than twice the distance between the line l and m) then the equidistant curves intersect. This is easy to see in the upper half-plane.

(b) Right: in the upper half-plane we can represent the two parallel lines by vertical rays and the corresponding distance d equidistant curves will be represented by two sets of parallel rays (at equal angle to the vertical line). This will produce one intersection.

(c) Right: There will be actually 4 intersection points: every of the two branches of e will intersect every of the two branches of f.

(d) Wrong in view of the result in (c).

Answer:(b), (c).

2.9 Survey, Week 19: Discrete group actions

Feedback to Survey, Week 19

There ware two questions:

Question 1. *Which of the following statements are true? (list all) Isometries of hyperbolic plane act transitively on ...*

(a) pairs of horocycles;

(b) pairs of tangent horocycles;

(c) triples of mutually tangent horocycles;

(d) pairs of tangent equidistant curves to different lines.

Statistics: unfortunately, I am unable to view your answers as DUO does not show at the moment any information about the results of the surveys...

Solution: (a) False: Two horocycles may have the same centre or different centres.

(b) True: In the upper half-plane, we can map any horocycles to the one represented by the horizontal line $y = 1$. Then, after parabolic transformation $z \to z + a$ every horocycle tangent to the first one coincides with $|z - i/2| = 1/2$.

(c) True: Any horocycle tangent to both horocycles described in (b) is $|z-i/2\pm 1| = 1/2$, after parabolic translation $z \to z + 1$ (if needed) we can assume that we have the two horocycles from (b) and $|z - i/2 - 1| = 1/2$

(d) False: For every two ultra-parallel lines one can construct a pair of equidistant curves to them tangent to each other. Notice that the ends of the equidistant curves allow us to reconstruct the lines, and hence, one can reconstruct the distance between the lines, which can be any positive number (and which should be preserved by isometries).

Answer: (b), (c).