Solutions 11-12

11.1 Show that Möbius transformations form a group.

Solution. We need to prove 4 properties of a group:

1) composition of Möbius transformations is a Möbius transformation:

$$g(f(z)) = \frac{a_2\frac{a_1z+b_1}{c_1z+d_1}+b_2}{c_2\frac{a_1z+b_1}{c_1z+d_1}+d_2} = \frac{(a_2a_1+b_2c_1)z+(a_2b_1+b_2d_1)}{(c_2a_1+d_2c_1)z+(c_2b_1+d_2d_1)}.$$

We also need to check the condition $ad-bc\neq 0$ for $g(f(z))$. Computing $ad-bc$ we get 8 terms, four of which cancel, and four other can be regrouped to

$$(a_1d_1-b_1c_1)(a_2d_2-b_2c_2),$$

which is non-zero since the multiples are non-zero.

2) existence of **identity** map in the set of Möbius transformation:

$$f_{id}(z) = \frac{z+0}{0\cdot z+1} = z$$

3) existence of **inverse** map in the set of Möbius transformation: we find the inverses for $f_1(z) = az+b$ and $f_2(z) = 1/z$:

$$f_1^{-1}(z) = \frac{1}{a}z - \frac{b}{a} \qquad f_2^{(-1)}(z) = \frac{1}{z}.$$

It was shown in the lecture that every Möbius transformation may be obtained as a composition of several transformations of type f_1 and f_2. The inverse of the composition is the composition of inverses $(g_k\circ\cdots\circ g_1)^{-1} = g_1^{-1}\circ\cdots\circ g_k^{-1}$.

4) associativity: proved in the lecture.

Remark: we demonstrate the properties by the direct computations, but of course one can use instead the same reasoning as in the lecture, i.e. modelling the action of $f(z)=\frac{az+b}{cz+d}$ by multiplication by matrix $\begin{pmatrix} a & b \\ c & d\end{pmatrix}$. Then identity map corresponds to identity matrix, inverse map corresponds to inverse matrix (lies in the group $GL(2,\mathbb{C})$!), associativity is associativity of matrix multiplication and closure under composition is the closure of $GL(2,\mathbb{C})$ under multiplication of matrices.

Shortly speaking, all group properties of Möbius transformations follow from the corresponding group properties of $GL(2,\mathbb{C})$.

11.2 Find a Möbius transformation which takes $1, 2, 3$ to $0, 1, \infty$.

Solution. Let $f(z) = \frac{az+b}{cz+d}$.
Since $f(1) = 0$ we see $b = -a$.
Since $f(3) = \infty$, we have $d = -3c$ (and we can also assume $c = 1$ as c can not be zero).
Finally, since $f(2) = 1$ we have $f(2) = \frac{a(2-1)}{2-3} = -a = 1$.
So, $f(z) = \frac{-z+1}{z-3}$. (Notice that $ad-bc = 3-1\neq 0$, so it is a Möbius transformation.)

11.3 (*)

(a) Let l be a line and γ be a circle. Show that γ is orthogonal to l if and only if l contains the centre of γ.

(b) Let γ_1, γ_2, γ_3 be three mutually orthogonal circles on the plane. Show that there exists a Möbius transformation which takes them to the curves $\{x = 0\}$, $\{y = 0\}$ and $x^2 + y^2 = 1$.

Solution. (a) The line l through the centre O of the circle is orthogonal to the circle as the radius is orthogonal to the tangent line. In more details, the reflection r_l with respect to l preserves the circle since $O \in l$, so, it preserve the intersection point of the circle with l, and hence, preserves the tangent at this point (by uniqueness of the tangent). This implies that the tangent is orthogonal to l.

Now, consider a line not through O intersecting the circle at points A amd B. As it was shown above, the radius OA is orthogonal to the tangent, which implies that BA is not orthogonal to it (as $O \notin AB$).

(b) First, we map (by a Möbius transformation) two intersections of the circles to 0 and ∞ (this is possible by triple transitivity of the group). This maps the two circles into two perpendicular lines (as Möbius transformations preserve angles). Then we apply a Euclidean isometry (composition of a rotation and translation, i.e. a Möbius transformation) to move these lines to the coordinate axes. The third circle is mapped to a circle or line orthogonal to both axes. This is clearly impossible for a line. For a circle this is only possible when O is the centre of the circle (here we use (a)). Finally, applying $f(x) = kz$ with $k \in \mathbb{R}$ if needed, we can make sure that the third curve is the unit circle.

11.4 (*) Let γ be a circle and P be a point lying outside of γ. Let l be a line through P and A, B be the intersection points of l with γ. Prove that the product $|PA| \cdot |PB|$ does not depend on the choice of l. (This product is also called *power of P with respect to γ*).

Solution. Let $l_1 \neq l$ be another line though P and let A_1, B_1 be the intersection points of l_1 with γ (see Figure (a) below). The quadrilateral ABB_1A_1 is inscribed into γ, so by E29,

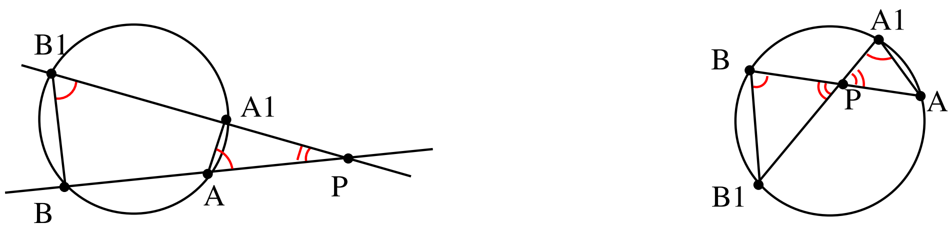
$$\angle BB_1A_1 + \angle BAA_1 = \pi,$$

which implies $\angle BB_1P = \angle A_1AP$. Hence, the triangle B_1BP is similar to the triangle AA_1P (by two angles: they have the common angle P and a pair of equal angles as above). Therefore,

$$\frac{PA}{PB_1} = \frac{PA_1}{PB},$$

which implies

$$PA \cdot PB = PA_1 \cdot PB_1.$$



(a) (b)

11.5 The same question as 10.1, but P lies inside γ.

Solution. This is quite similar to above (see Figure (b)). This time the triangles B_1BP and AA_1P are similar again by two angles (angles at P are vertical; $\angle B_1BP = \angle AA_1P$ as angles in the same circular segment, E28), and we get exactly the same equalities as above.

12.1 Prove the theorem of Ptolemy: for a cyclic quadrilateral $ABCD$, the following equality holds:

$$AB \cdot CD + BC \cdot AD = AC \cdot BD.$$

Solution. First, we divide by $AC \cdot BC$, so that we need to prove

$$\frac{AB \cdot CD}{AC \cdot BD} + \frac{AD \cdot BC}{AC \cdot BD} = 1.$$

Rewriting products as ratios

$$\frac{AB}{AC}/\frac{DB}{DC}+\frac{AD}{AC}/\frac{BD}{BC}=1, \tag{1}$$

we easily recognize (modules of) two cross-ratios $[B,C,A,D]$ and $[D,C,A,B]$ on the left.

Here, we understand A,B,C,D as complex numbers and AB as $B-A$. The only problem is that in general cross-ratio is also a complex number, but in our equation we have a positive real numbers. Since A,B,C,D lie on a circle, we know that the cross-ratio is real. We can also see that the cross-ratios are positive: for that we can map the circle to a real line by a Möbius transformation f (i.e. by a transformation preserving cross-ratios), so that the points $f(A)<f(B)<f(C)<f(D)$; then the positivity of the cross-ratios above easily follows.

So, we see that in the case of a cyclic quadrilateral we can understand equation (1) as $[B,C,A,D]+[D,C,A,B]=1$. Now, we notice that the second cross-ratio is obtained from the first one by permutation of the points and recall from Problem 7.8 that if $[B,C,A,D]=\lambda$ then $[D,C,A,B]=1-\lambda$.

12.2 (*) (Inversion with ruler and compass).

(a) Given a circle γ, construct its centre.

Solution. We take any three points A,B,C on the circle. Then the perpendicular bisectors of AB and AC pass through the centre of the circle, so the centre is just the intersection of two perpendicular bisectors (we know how to construct perpendicular bisectors from the problem class last term!).

(b) Given segments of length a and b construct a segment of length h satisfying $h^2=a\cdot b$.

Solution. We use a right-angled triangle ABC (with right angle C), as an altitude CH of a right angled triangle satisfies $CH^2=HA\cdot HB$ (see E23).

To construct this triangle we start from the points H and find the points A and B on one line (so that $AH=a$, $BH=b$ and H lies between A and B). To construct the point C, the vertex of the right angle, we will draw a circle γ_1 with diameter AB (centred at midpoint of AB) and find C as intersection of γ_1 with the line perpendicular to AB through H. Then $\angle ACB$ is right as it is an angle in a semicircle (E26), and CH is the altitude by construction. So we may apply E20 to see that the segment CH satisfies all required properties.

(c) Given a circle γ and a point P outside the circle, construct a line PQ tangent to γ.

Solution. Consider the tangent line PQ assuming that $Q\in\gamma$. Using the result of Question 11.4 we see that $PQ^2=PA\cdot PB$ where A,B are the intersection points of γ with arbitrary line l through P. We can easily construct an arbitrary line l through P and the intersection points A and B, so we get the segments PA and PB. So, by Part (b) we can construct the segment of length PQ, so that Q is the intersection of γ with the circle centred at P of radius PQ.

(d) Given a circle γ and a point A outside the circle, construct the inversion image of A

Solution. First, we construct the tangent line AQ (with $Q\in\gamma$) to the circle γ (we can do it by Part (c)).

Now, let A' be the orthogonal projection of Q to OA. We will prove that A' is the inversion image of A with respect to γ.

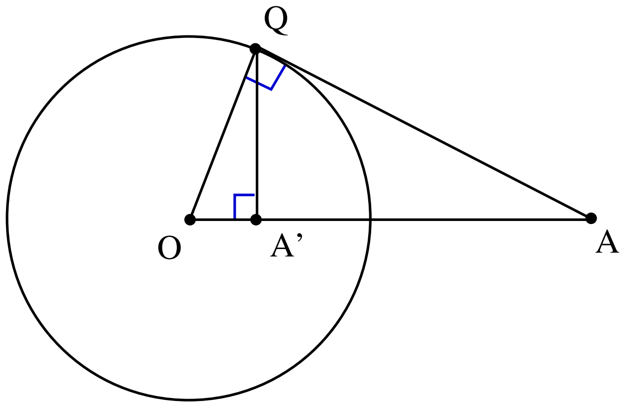
Notice that we can construct A': by Part (a) we can construct the centre O of γ, then we know how to construct the orthogonal projection of a given point (Q) to a given line (OA) since we did that in the problems class.

To prove that A' is the inversion image of A with respect to γ, consider the right triangles AOQ and QOA' (the angle $\angle QA'A$ is right by construction and the angle $\angle OQA$ is right as it is an angle between the radius and the tangent). These triangles also have a common angle O, so they are similar. Therefore,

$$\frac{OA}{OQ}=\frac{OQ}{OA'},$$

which implies $OA \cdot OA' = R^2$ where $R = OQ$ is the radius of γ.

Remark: To prove that the angle between a radius and the corresponding tangent is right, one may consider the reflection with respect to the radius. This reflection preserved the circle, so it should preserve the tangent, so the tangent should be orthogonal to the radius).



(e) Construct the inversion image for the point A' lying inside the circle γ.

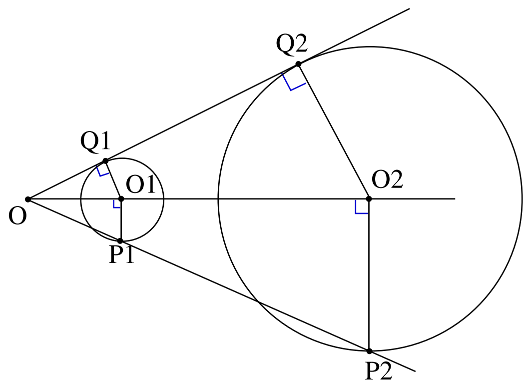
Solution. We invert the construction in (d): first we find Q as an intersection of γ with the line through A' perpendicular to OA, then draw the tangent line at Q (it is orthogonal to the radius OQ) and find A as the intersection with OA.

(f) Let O, A' and A be three points lying on a line (A' lies between O and A).
Construct a circle γ centred at O such that the inversion with respect to γ takes A to A'.

Solution. This is similar to Part (b) (with $a = OA$, $b = OA'$ and the unknown radius $R = h$).

(g) Given two circles γ_1 and γ_2, construct a line tangent to both of them.

Solution. First, suppose that the required tangent line l is already constructed. Denote by O_1 and O_2 the centres of the two circles, denote also $Q_1 = l \cap \gamma_1$ and $Q_2 = l \cap \gamma_2$. Let O be the intersection point of l with O_1O_2 (suppose also that l is not parallel to O_1O_2). Then the circle γ_2 may be obtained from γ_1 by a homothety with centre O and coefficient OO_2/OO_1 (indeed, the triangles OO_1Q_1 and OO_2Q_2 are similar by two angles as both have a right angle and have a common angle O, so $OO_2/O_1Q_1 = OQ_2/O_2Q_2$) In particular, if $P_1 \in \gamma_1$ and $P_2 \in \gamma_2$ are the farthest points points of γ_1 and γ_2 from the line O_1O_2, then O lies on the line P_1P_2.



We can use the consideration above to construct O. Indeed, it is easy to construct the points P_1 and P_2 ($P_i = \gamma_i \cap l_i$, where l_i is the line through O_i perpendicular to O_1O_2). Then we find O as the intersection of P_1P_2 with O_1O_2. Finally, we construct $Q_1Q_2 = OQ_1$ as in Part (c).

If the line P_1P_2 is parallel to O_1O_2, then P_1P_2 is the required tangent line.

(h) Given two circles γ_1 and γ_2 of different sizes, construct an inversion which takes γ_1 to γ_2 and takes γ_2 to γ_1.
(You need to construct the centre and the radius of the circle of inversion).

Solution. Let O_1 and O_2 be the centres of the circles and let $O = O_1O_2 \cap l$ be the intersection of O_1O_2 with the line l tangent to both circles (constructed as in (g)). Let $l \cap \gamma_1 = Q_1$, $l \cap \gamma_2 = Q_2$ be the intersection points. Let $H \in O_1O_2$ be a point such that $OH^2 = OQ_1 \cdot OQ_2$ (constructed as in (b)). Let γ be the circle centred at O of radius OH. We will prove that the inversion in γ swaps the circles γ_1 and γ_2.

Denote by I_γ the inversion in γ. As $OH^2 = OQ_1 \cdot OQ_2$ we see that $I_\gamma(Q_1) = Q_2$ and $I_\gamma(Q_2) = Q_1$. Furthermore, I_γ takes the tangent line Q_1Q_2 to itself (as it passes through O) as well as it takes to itself the other tangent line m (another line through O tangent to both circles). So, I_γ should take the circle γ to a circle tangent to l at Q_2 and also tangent to m (since inversion takes circles not through the origin to circles and preserves angles, and since $I_\gamma(Q_1) = Q_2$). It is easy to see that γ_2 is the only circle satisfying these conditions, so, $I_\gamma(\gamma_1) = \gamma_2$. The same reasoning shows that $I_\gamma(\gamma_2) = \gamma_1$.

(i) Given two circles γ_1 and γ_2 of different sizes,
find an inversion which takes them to a pair of equal circles.
(You need to construct the centre and the radius of the circle of inversion).

Solution. Let γ_0 be the circle such that the inversion with respect to γ_0 swaps γ_1 and γ_2 (constructible as in (h)).

Denote by O the intersection point of γ_0 and O_1O_2 not lying between O_1 and O_2, denote by H the other intersection. Consider an inversion I with respect to a circle γ centred at O of radius OH (both easily constructible!).

As γ_0 passes through O, I takes γ_0 to a straight line l_0. As neither of γ_1 and γ_2 passes through O, both $I(\gamma_1)$ and $I(\gamma_2)$ are circles. Let r_0 be the reflection with respect to l_0. We will show that r_0 swaps the circles $I(\gamma_1)$ and $I(\gamma_2)$. Then we conclude that the circles are equal.

To show that r swaps the circles $I(\gamma_1)$ and $I(\gamma_2)$, consider the composition $f = I \circ I_{\gamma_0} \circ I$. Clearly, f takes l_0 to itself pointwise (I takes a point on l_0 to a point on γ_0, then I_{γ_0} preserves it and I takes it back to initial place). Also, as a composition of inversions, it should take lines and circles to lines and circles. Since the infinite point is preserved by f (lying on l_0), f takes lines to lines. So, by Theorem 3.7. (Fundamental Theorem of affine geometry) f is an affine map. Also, as a composition of inversions, f preserves angles. So, f is a similarity map. Furthermore, as f preserves all points of l_0, f is an isometry. Finally, as f changes the orientation, f is the reflection r_0 with respect to the line l_0:

$$r_0 = I \circ I_{\gamma_0} \circ I.$$

Now,

$$r_0(I(\gamma_1)) = I \circ I_{\gamma_0} \circ I(I(\gamma_1)) = I \circ I_{\gamma_0}(\gamma_1) = I(\gamma_2),$$

and similarly,

$$r_0(I(\gamma_2)) = I(\gamma_1).$$

So, the circles $I(\gamma_1)$ and $I(\gamma_2)$ are of the same size.

12.3 What type is the transformation $1/z$?
(Hint: parabolic or not? if not, then is it elliptic, or hyperbolic, or loxodromic?)

Solution. The transformation $f(z) = 1/z$ fixes the points $z = 1$ and $z = -1$, so f is not parabolic. $f^2 = Id$, so the fixed points are not attracting or repelling. So, $1/z$ is elliptic.

12.4 Write the following transformations as compositions of inversions and/or reflections:
(a) $2z$ (b) $-z$ (c) $z + 1$ (d) $\frac{1}{z}$

Solution. We will present each of the transformations as a composition of two inversions/reflections $f(z) = f_2 \circ f_1(z)$. Denote by s_1 and s_2 the fixed set (i.e. circle or line) of f_1 and f_2 respectively.

(a) s_1 and s_2 are circles $|z| = 1$ and $|z| = \sqrt{2}$; $f_1(z) = 1/\bar{z}$, $f_2(z) = 2/\bar{z}$.

(b) s_1 and s_2 are lines $Im(z) = 0$ and $Re(z) = 0$; $f_1(z) = \bar{z}$, $f_2(z) = -\bar{z}$.

(c) s_1 and s_2 are line $Re(z) = 0$ and $Re(z) = 1/2$; $f_1(z) = -\bar{z}$, $f_2(z) = -\overline{z - \frac{1}{2}} + \frac{1}{2} = -\bar{z} + 1$.

(d) s_1 is the real line, s_2 is the unit circle. $f_1 = \bar{z}$, $f_2 = \frac{1}{\bar{z}}$.

12.5 Let I be an inversion with respect to the unit circle $|z| = 1$. Find the image $I(l)$ of the line l given by the equation $Re(z) = 2$.

Solution. l passes through infinity and doesn't pass through the origin, so $I(l)$ is a circle through the origin. l passes through $z = 2$, so $L(l)$ passes through $z = 1/2$. Both l and the unit circle are symmetric with respect to the line $y = 0$, so, $I(l)$ is symmetric with respect to the line $y = 0$. (In other words, I preserves this line and takes the line orthogonal to it to a circle ortogonal to it). Hence, $I(l)$ is the circle $(x - \frac{1}{4})^2 + y^2 = \frac{1}{16}$.

12.6 Do the points $-1 - 2i$, $-1 + 2i$, $3 + i$, $3 - i$ lie on one line or circle?

Solution. Four points lie on the same line or circle if their cross-ratio is real.
$[-1 - 2i, -1 + 2i, 3 + i, 3 - i] = \frac{(3+i+1+2i}{3+i-(-1+2i)} / \frac{3-i+1+2i}{3-i-(-1+2i)} = \frac{4+3i}{4-i} \frac{4-3i}{4+i} \in \mathbb{R}$
Hence, these points lie on one circle or line.

12.7 Show that a finite order Möbius transformation is elliptic.
(g is called of finite order if $g^n = id$ for some integer n).

Solution. First, notice that a conjugation preserve the order of the transformation.

A parabolic Möbius transformation is conjugate to $z+1$. This transformation is of infinite order, so a finite order Möbius transformation is not parabolic.

A non-parabolic Möbius transformation is conjugate to αz, $\alpha \in \mathbb{C}$, $\alpha \neq 0$. If $|\alpha| \neq 1$ than $|z^n|$ growth (or decreases) when n tends to infinity. So for a finite iorder Möbius transformation we have $|\alpha| = 1$. Hence, it is elliptic.

12.8 Find a parabolic Möbius transformation preserving the point $z = 1$.

Solution. Let $f(z) = z + 1$, it is a parabolic transformation preserving ∞. Let g be a transformation which takes 1 to ∞, say $g = \frac{1}{z-1}$. Then $\phi = g^{-1} \circ f \circ g$ preserves $z = 1$.

To find g^{-1} notice that $g = g_2 \circ g_1$ where $g_1(z) = z - 1$, $g_2(z) = 1/z$. Since $g_1^{-1} = z + 1$, $g_2^{-1} = 1/z$, we have $g^{-1} = g_1^{-1} \circ g_2^{-1} = \frac{1}{z} + 1$.
Hence, $\phi = g^{-1} \circ f \circ g = \frac{1}{\frac{1}{z-1}+1} = \frac{z-1}{1+z-1} + 1 = \frac{z-1}{z} + 1 = \frac{-1}{z} + 2$ is a parabolic transformation preserving $z = 1$.

12.9 Find a Möbius transformation mapping the disc $|z| < 1$ to the half-plane $Rez > 2$.

Solution.

First, we will find a transformation $g(z) = \frac{az+b}{cz+d}$ mapping the disc to the upper half-plane. More precisely, we will map $1, i, -1$ to $0, 1, \infty$:

> Since $g(1) = 0$ we have $a + b = 0$, so $b = -a$. As $g \neq 0$ we may assume $a \neq 0$, and we may assume $a = 1$ (after dividing all of a, b, c, d by the same number).
> Since $g(-1) = \infty$, we have $-c + d = 0$, so $d = c$.
> Since $g(i) = 1$, we have $\frac{i-1}{ci+c} = 1$, which implies $c = \frac{i-1}{i+1} = i$. Hence, $g(z) = \frac{1}{i} \frac{z-1}{z+1} = -i \frac{z-1}{z+1}$.

Since g takes $1, i, -1$ to $0, 1, \infty$, it takes the unit circle to the reals.
Since $g(0) = -i(-1) = i$, the disc is mapped to the upper half-plane.

Now, we compose $g(z)$ with rotation by $\pi/2$ clockwise, i.e. multiplication by $-i$, followed by translation by 2 (i.e. adding 2). So, we get

$$f(z) = -ig(z) + 2 = (-i) \cdot (-i) \frac{z-1}{z+1} + 2 = -\frac{z-1}{z+1} + 2 = \frac{z+3}{z+1}.$$

12.10 I_0 is the inversion with respect to the circle $|z| = 1$. I_1 is the inversion with respect to the circle $|z-1| = 1$. The composition $I_1 \circ I_0$ is a Möbius transformation. What type is the composition $I_1 \circ I_0$?
(Hint: try to find a geometric solution, without writing the formulas).

Solution. The circles $|z| = 1$ and $|z-1| = 1$ have two points of intersection. Both of these two points are fixed points of f. So, f is not parabolic.

Let as check if the fixpoints of f are repellent/attractive. Consider the orbit of the point ∞:

$$\infty \xrightarrow{I_0} 0 \xrightarrow{I_1} 0 \xrightarrow{I_0} \infty \xrightarrow{I_1} 1 \xrightarrow{I_0} 1 \xrightarrow{I_1} \infty$$

This implies that $(I_1 \circ I_0)^n(\infty)$ is 0,1 or ∞ for any n, so that it does not tend to any of the fixed points of f. Hence, the fixed points are not attractive or repellent. So, $f = I_1 \circ I_0$ is an elliptic transformation.

Remark: we can also see that f is or order 3, since f^3 preserves ∞ and two intersection points of the circles.