Solutions 3-4

3.1 Let ABC be a triangle and let f be an isometry. Prove that the points C and D lie on the same side with respect to the line AB if and only if the points $f(C)$ and $f(D)$ lie on the same side with respect to the line $f(A)f(B)$.

Solution: Suppose that C and D lie on the same side with respect to the line AB but $f(C)$ and $f(D)$ lie on different sides with respect to the line $f(AB)$. Then the segment CD does not intersect the line AB while the segment $f(C)f(D)$ does intersect the line $f(AB)$. In particular,

$$0 < \min_{x\in AB,\ y\in[CD]} d(x,y) = \min_{x'\in f(AB),\ y'\in f([C,D])} d(x',y') = 0,$$

which is impossible.

Similarly, one can show the impossibility of the assumption that C and D lie on different sides with respect to AC, while $f(C)$ and $f(D)$ lies on the same side with respect to $f(AB)$.

3.2 Let $f, g \in Isom(\mathbb{E}^2)$. Show that $\quad g(x) \in Fix_{gfg^{-1}} \Leftrightarrow x \in Fix_f \qquad$ for all $x \in \mathbb{E}^2$.

Solution: Suppose first that $x \in Fix_f$, i.e. $f(x) = x$. Then $g \circ f \circ g^{-1}(g(x)) = g \circ f(x) = g(x)$, i.e. $g(x) \in Fix_{gfg^{-1}}$.

Similarly, if $g(x) \in Fix_{gfg^{-1}}$ then $g(x) = gfg^{-1}(g(x))$. Also $gfg^{-1}(g(x)) = g(f(x))$. So, we get that $g(x) = g(f(x))$. As g is a bijection, this implies $x = f(x)$.

3.3 (*) Show that the map

$$f(\mathbf{x}) = A\mathbf{x}, \qquad A \in GL_2(\mathbb{R}),$$

is an isometry if and only if $A \in O_2(\mathbb{R})$ (i.e. $A \in GL_2(\mathbb{R})$, $A^T A = I$).

Solution: Suppose that $f = A\mathbf{x}$ is isometry, $A \in GL_2(\mathbb{R})$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$$\sqrt{x_1^2 + x_2^2} = d(\mathbf{x}, \mathbf{0}) = d(f(\mathbf{x}), f(\mathbf{0})) = \sqrt{(ax_1 + bx_2)^2 + (cx_1 + dx_2)^2} =$$
$$= \sqrt{(a^2 + c^2)x_1^2 + (c^2 + d^2)x_2^2 + 2(ab + cd)x_1x_2}$$

should hold for any choice of $\mathbf{x} = (x_1, x_2)$. In particular, the choice $(x_1, x_2) = (1, 0)$ gives $a^2 + c^2 = 1$, the pair $(0, 1)$ gives $b^2 + d^2 = 1$, and the pair $(1, 1)$ gives $ab + cd = 0$. This is exactly the condition that $A^t A = I$:

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix}.$$

So, if $f(\mathbf{x}) = A\mathbf{x}$ is an isometry then $A \in O_2(\mathbb{R})$.

Suppose now that $A \in O_2(\mathbb{R})$. We need to show that $f(\mathbf{x}) = A\mathbf{x}$ preserves the distance from x to any point y. For the point $\mathbf{y} = (0, 0)$ the equality $d(\mathbf{x}, \mathbf{y}) = d(f(\mathbf{x}), f(\mathbf{y}))$ is exactly the above computation (read backwards). For arbitrary point $\mathbf{y} = (y_1, y_2)$ it is the same computation with x_1 substituted by $(x_1 - y_1)$ and x_2 substituted by $(x_2 - y_2)$.

3.4 Let $f : z \mapsto 2z$, $z \in \mathbb{C}$. Let G be a group of transformations of $\mathbb{E}^2$ generated by f.

(a) Does G act discretely on $\mathbb{C}$? Justify your answer.

(b) Show that G acts discretely on $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

(c) Find a fundamental domain for the action $G : \mathbb{C}^*$.

Solution: (a) G does not act descretely on $\mathbb{C}$ as the point 0 is an accumalation point for each orbit (for example, the orbit of the point $z = 1$ consists of all points $2^n \in \mathbb{C}$, $n \in \mathbb{Z}$, in particular, the sequence $1/2^n$ tends to 0).

(b) An orbit of a point $z \in \mathbb{C}$ consists of all points $2^n z \in \mathbb{C}$, this set has no accumulation points in $\mathbb{C} \setminus \{0\}$ (for each point of the orbit one can find a ball neighbourhood containing no other points of the same orbit - what could be the radius of such a ball?)

(c) one possible answer is to take an annulus $\{z \in \mathbb{C} \mid 1 < |z| < 2\}$. Then it is easy to see that the G-images of this annulus tile $\mathbb{C}^*$ without intersections. Can a fundamental domain be of some other shape?

3.5 (*) Let P be a regular hexagon on $\mathbb{E}^2$.

(a) Find a group H acting on $\mathbb{E}^2$ discretely and such that P is a fundamental domain for the action $H : \mathbb{E}^2$. (Describe the group in terms of its generators).

(b) Let G be a group generated by reflections with respect to the sides of P. Show that G is discrete.

(c) Find a fundamental domain for G.

(d) Is H a subgroup of G? If yes, find its index $[G : H]$.

(e) describe the orbit space of the action $H : \mathbb{E}^2$.
Hint: if you were not too creative in part (a) you will probably get some space we already met in the course.

Solution: (a) The easiest way is to define the group as a group generated by two translations. More precisely, consider a tiling of the plane by regular hexagons. Let $\mathbf{u}$ and $\mathbf{v}$ be the vectors from the center of P to the centers of two its neighbours and such that $\mathbf{u} \neq \mathbf{v}$. Then we can take the group generated by the translations T_u and T_v by these two vectors.

Notice that both T_u and T_v preserve the hexagonal tiling of the plane, so each element preserves it. Also, all elements of the group are isometries.

We need to check that each point $p \in \mathbb{E}^2$ has a neighbourhood containing at most finitely many points of each orbit. Notice that

- there are finitely many isometries of $\mathbb{E}^2$ taking a regular hexagon to itself.
- there are finitely many hexagons gP, $g \in G$ intersecting any given disc on $\mathbb{E}^2$.

So, every disc contains finitely many points of each orbit, and hence, H acts discretely on $\mathbb{E}^2$.

Now, we need to prove that P is the fundamental domain of the action. It is easy to see that the hexangond hP, $h \in H$ cover the whole plane. Moreover, there is exactly one element of H taking P to any other hexagon of the hexagonal tiling (translation by the vector connecting the centres), so each hexagon of the tiling only contains one point of each orbit. Finally, every point on a boundary of the hexagon P belongs to at most 3 hexagons. So, P is a fundamental domain for the action $H : \mathbb{E}^2$.

(b) All generators of the group G preserve the hexagonal tiling of the plane, so, all elements of G do preserve it. Hence, by the same reason as above in (a) we conclude that G is acting discretely on $\mathbb{E}^2$.

(c) Let as subdivide each hexagon in the tesselation into 6 regular triangles. Then every generator of G preserves the tiling of the plane by the triangles. In principle, there are many isometries taking a regular triangle to itself, however inside the group G we have some constrains:

(α) every element of G maps the centre of a hexagon to a centre of (another) hexagon;

(β) we can color the triangles in the alternating way black and white. Then every reflection takes all black triangles to white ones and all white triangles to the black one. This implies that if T is white triangle and $gT = T$ for some $g \in G$ then g is orientation-preserving. Combining this with observation (α) above we see that $g = Id$ (it is orientation preserving isometry, taking T to itself and taking one vertex of T to itself).

As the copies gT, $g \in T$ cover the plane, we conclude that a regular triangle T is a fundamental domain of $G : \mathbb{E}^2$.

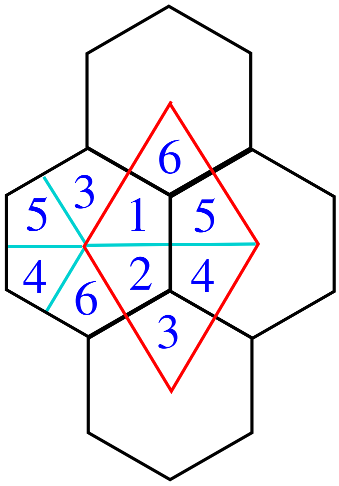
(d) Yes, H is a subgroup of G. To see that we need to show that both translations generating H lie in G. We will show it just for one translation u - the other one is similar.

First, we prove that G contains reflections with respect to all sides of all hexagons in the tiling. It is sufficient to see that for the sides of one tile adjacent to P (then we can move to any given tile in several steps). Let $r_1, \ldots, r_6$ be the reflections with respect to the sides of P. and let $r_1(P)$ be the adjacent tile. Then the reflections with respect to the sides of $r_1(P)$ may be written as $r_1, r_1r_2r_1, r_1r_3r_1, \ldots, r_1r_6r_1$ (check it!).

Now, let T_u is the translation by u and suppose that $T_u(P)$ coincide with $r_1(T)$. Then T_u is a composition of two reflections with respect to two parallel lines, namely r_1 and a reflection with respect to the diagonal of the hexagon. Both reflections lie in G as we have seen in the paragraph above.

The index $[G : H]$ is the number of (left) cosets of H in G, which may be found as the number of fundamental domains of the group G in one fundamental domain of the subgroup H (here, it is 6 triangles in a hexagon). This counts how many different elements of G are there modulo action of H.

(e) The orbit space of $H : \mathbb{E}^2$ is a torus (the corresponding latice is generated by the translations by u and v). One can also see it by following cutting and pasting procedure: let Q be a quadrilateral with one vertex in the center of the hexagon, one in a vertex and two vertices in the midpoints of its sides (six copies of q tile the hexagon). Then the six quadrilaterals may be rearranged into a parallelogram is in the following figure:



Remark. The quadrilateral Q is not a fundamental domain for $G : \mathbb{E}^2$ as there is a reflection in G which takes Q to itself.

4.1 Let $G : \mathbb{E}^2$ be a cyclic group generated by a translation T. Let X be an orbit space of $G : \mathbb{E}^2$.

(a) Show that X is an infinitely long cylinder which admits a Euclidean metric (i.e. each point on X has a neighbourhood isometric to a domain in $\mathbb{E}^2$).

(b) Find a closed geodesic on X;

(c) Find an open geodesic on X.

Solution: (a) Let F be a fundamental domain for the action $G : \mathbb{E}^2$ which looks like an infinite strip (in particular, we can take a Dirichlet domain). Then X is an infinite strip F with opposite sides identified. It is clear that each "inner" point for the strip has a good neighbourhood. For a "boundary point" of the strip we glue the disc neibourhood from two semi-disks. So, X inherits Euclidean structure from $\mathbb{E}^2$.

(b) Let $\mathbf{x} \in \mathbb{E}^2$ be any point and let $T\mathbf{x}$ be its image under the translation. Consider the line l through $\mathbf{x}$ and $T\mathbf{x}$. Then $\hat{l} = l/G$ (the set of points of l modulo action of G) is a closed geodesic.

(c) Let l_1 be a line orthogonal to the direction of translation. Then l_1 is contained entirely in F, so $\hat{l}_1 = l_1/G$ is again a line. Then $\hat{l}$ is an open geodesic in X.

4.2 Let X be a torus obtained by identification of opposite sides of the Euclidean square.

(a) Are there closed geodesics on X?;

(b) Are there open ones?

Solution: (a) Let the square be a square with side 1, with the vertices lying in the integer lattice $L = \{(x_1, x_2)|x_1, x_2 \in \mathbb{Z}\}$ and sides parallel to the coordinate axes. Let $l \in \mathbb{E}^2$ be a line passing through two distinct points of the lattice L and let $\hat{l} = l/G$ where G is a group generated by unit translations along the coordinate axes (so that $X = \mathbb{E}^2/G$). Then $\hat{l}$ is a closed geodesic on X (it closes when it first comes to an integer point).

(b) Let l be a line through $O = (0,0)$ which makes with the axis Ox an angle θ with irrational $tan\theta$. Then l never passes through any integer point $(x_1, x_2) \in L$ except for $(0,0)$. Let $\hat{l} = l/G$. Then no points of l will be identified, so $\hat{l}$ is an open geodesic on X.

4.3 (*) Given ruler and compass and a circle $\mathcal{C}$ on the plane, construct the centre of the circle. You can use without proofs and further descriptions the construction of perpendicular bisector for a given segment.

Solution: Let A and B be two points on the circle $\mathcal{C}$. Let l_{AB} be the perpendicular bisector for AB. Then the centre O of the circle lies on l_{AB} (as it lies on the same distance from A and B). Similarly, for a point $D \in \mathcal{C}$ we can also look at the segment AD, its perpendicular bisector l_{AD} also contains O. So, $O = l_{AB} \cap l_{AD}$.

4.4 (*) Does there exist a map of a domain on the sphere onto a domain on the Euclidean plane that takes the segments of spherical lines into segments of Euclidean lines?

Solution: Yes! Below we construct such a map as a projection of a triangle on the sphere to a plane. Let ABC be a spherical triangle. Consider a Euclidean plane in $\mathbb{E}^3$ through the points A, B, C. Consider a projection φ of the sperical triangle ABC from the center O of the sphere to the plane ABC. It will take the sperical triangle ABC to the Euclidean triangle ABC. Furthermore, a part of a greate circle through MN on the sphere will be mapped to the intersection of the plane ABC with the plane OMN, which is a line on the plane.

Note, that we actually don't need this triangle ABC here at all: we could do the same for an open half-sphere (projecting it to the whole plane).