Assignment 15-16

Starred problems due on Friday, 1 March

15.1. (*)

(a) Let P and Q be feet of the altitudes in an ideal hyperbolic triangle. Find PQ.

(b) Find the radius of a circle inscribed into an ideal hyperbolic triangle.

(c) Show that a radius of a circle inscribed into a hyperbolic triangle does not exceed $arcosh(2/\sqrt{3})$.

15.2. For a right hyperbolic triangle ($\gamma = \frac{\pi}{2}$) show:

$$\text{(a)} \tanh b = \tanh c \cos\alpha, \qquad \text{(b)} \sinh a = \sinh c \sin\alpha.$$

15.3. Show that in the upper half-plane model the following distance formula holds:

$$2\sinh^2 \frac{d}{2} = \frac{|z-w|^2}{2Im(z)Im(w)}$$

.

15.4. Find an area of a right-angled hyperbolic pentagon.

15.5. In the upper half-plane model, find the locus of points z lying on distance d from the line 0∞.

16.1. In the Klein disc model draw two parallel lines, two ultra-parallel lines, an ideal triangle, a triangle with angles $(0, \frac{\pi}{2}, \frac{\pi}{3})$.

16.2. (*) Show that three altitudes of a hyperbolic triangle either have a common point or are pairwise parallel or there is a unique line orthogonal to all three altitudes.

16.3. Let u, v be two vectors in $\mathbb{R}^{2,1}$. Denote $Q = |\frac{\langle u,v\rangle^2}{\langle u,u\rangle\langle v,v\rangle}|$, where $\langle x, y\rangle = x_1y_1 + x_2y_2 - x_3y_3$. Show the following distance formulae:

(a) if $\langle u,u\rangle < 0$, $\langle v,v\rangle < 0$, then u and v give two points in $\mathbb{H}^2$, and $\cosh^2(d(u,v)) = Q$.

(b) if $\langle u,u\rangle < 0$, $\langle v,v\rangle > 0$, then u gives a point and v give a line l_v on $\mathbb{H}^2$, and $\sinh^2 d(u, l_v) = Q$.

(c) if $\langle u,u\rangle > 0$, $\langle v,v\rangle > 0$ then u and v define two lines l_u and l_v on $\mathbb{H}^2$ and

- if $Q < 1$, then l_u intersects l_v forming the angle φ satisfying $Q = \cos^2\varphi$;
- if $Q = 1$, then l_u is parallel to l_v;
- if $Q > 1$, then l_u and l_v are ultra-parallel lines satisfying $Q = \cosh^2 d(l_u, l_v)$.

16.4. (*) Consider the two-sheet hyperboloid model $\{u = (u_1, u_2, u_3) \in \mathbb{R}^{2,1} \mid \langle u,u\rangle = -1,\ u_3 > 0\}$, where $\langle u,u\rangle = u_1^2 + u_2^2 - u_3^2$.

(a) For the vectors

$$\begin{array}{lll} v_1 = (2,1,2) & v_2 = (0,1,2) & v_3 = (3,4,5) \\ v_4 = (1,0,0) & v_5 = (0,1,0) & v_6 = (1,1,2) \end{array}$$

decide if v_i corresponds to a point in $\mathbb{H}^2$, or a point in the absolute, or a line in $\mathbb{H}^2$.

(b) Find the distance between the two points of $\mathbb{H}^2$ described in (a).

(c) Which pair of the lines in (a) is intersecting? Which lines are parallel? Which are ultra-parallel?

(d) Find the distance between the pair of ultra-parallel lines in (a).

(e) Does any of the points in (a) lie on any of the three lines?

(f) Find the angle between the pair of intersecting lines.

References:

1. Lectures on elementary hyperbolic geometry, area, Klein model and hyperboloid model are based on Lectures VII, VIII, VI and XIII of Prasolov's book.
Alternatively, see Section 5.2 in Prasolov and Tikhomirov.