Assignment 3-4
Starred problems due on Friday, 3 November

3.1 Let ABC be a triangle and let f be an isometry. Prove that the points C and D lie on the same side with respect to the line AB if and only if the points $f(C)$ and $f(D)$ lie on the same side with respect to the line $f(A)f(B)$.

3.2 Let $f, g \in Isom(\mathbb{E}^2)$. Show that $g(x) \in Fix_{gfg^{-1}} \Leftrightarrow x \in Fix_f$ for all $x \in \mathbb{E}^2$.

3.3 (*) Show that the map
$$f(\mathbf{x}) = A\mathbf{x}, \qquad A \in GL_2(\mathbb{R}),$$
is an isometry if and only if $A \in O_2(\mathbb{R})$ (i.e. $A \in GL_2(\mathbb{R})$, $A^TA = I$).

3.4 Let $f : z \mapsto 2z$, $z \in \mathbb{C}$. Let G be a group of transformations of $\mathbb{E}^2$ generated by f.

(a) Does G act discretely on $\mathbb{C}$? Justify your answer.

(b) Show that G acts discretely on $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

(c) Find a fundamental domain for the action $G : \mathbb{C}^*$.

3.5 (*) Let P be a regular hexagon on $\mathbb{E}^2$.

(a) Find a group H acting on $\mathbb{E}^2$ discretely and such that P is a fundamental domain for the action $H : \mathbb{E}^2$. (Describe the group in terms of its generators).

(b) Let G be a group generated by reflections with respect to the sides of P. Show that G is discrete.

The following three subquestions are a bit more involved and are **not mandatory** for submission. You are very welcome to write down and show me your solution/sketches/ideas but also don't worry if you are not really sure how to do that.

(c) Find a fundamental domain for G.

(d) Is H a subgroup of G? If yes, find its index $[G : H]$.

(e) Describe the orbit space of the action $H : \mathbb{E}^2$.
Hint: if you were not too creative in part (a) you would probably get some space we have already seen in the course.

4.1 Let $G : \mathbb{E}^2$ be a cyclic group generated by a translation T. Let X be an orbit space of $G : \mathbb{E}^2$.

(a) Show that X is an infinitely long cylinder which admits a Euclidean metric (i.e. each point on X has a neighbourhood isometric to a domain in $\mathbb{E}^2$).

(b) Find a closed geodesic on X;

(c) Find an open geodesic on X.

4.2 Let X be a torus obtained by identification of opposite sides of the Euclidean square.

(a) Are there closed geodesics on X?

(b) Are there open ones?

4.3 (*) Given ruler and compass and a circle $\mathcal{C}$ on the plane, construct the centre of the circle. You can use without proofs and further descriptions the construction of perpendicular bisector for a given segment.

4.4 (*) Does there exist a map of a domain on the sphere onto a domain on the Euclidean plane that takes the segments of spherical lines into segments of Euclidean lines?

References:

1. For material of Lecture 5 look at
   - G. Jones, *Algebra and Geometry*, Lecture notes (Section 1).
2. You will find (almost) all material of Lecture 6 in
   - N. Peyerimhoff, *Geometry III/IV*, Lecture notes (Section 1.6).
3. Material of Lecture 7 may be found in Kiselev's Geometry / Book II. Stereometry
   - A. P. Kiselev, *Geometry / Book II. Stereometry.*
4. Starting from Lecture 8 we will follow
   - V. V. Prasolov, *Non-Euclidean Geometry*

   (our Lecture 8 is a first half of Prasolov's Lecture 1).