

Assignment 5-6
Starred problems due on Friday, 17 November

- 5.1 A circle $C_{A,r}$ of radius r centred at A is the set of points on distance r from A . Show that any spherical circle on a sphere $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ is represented by a Euclidean circle.
- 5.2 (*) Prove that in a spherical triangle (a) the perpendicular bisectors are concurrent; (b) the angle bisectors are concurrent.
- 5.3 Given SAS congruence law for spherical triangles, derive the ASA law.
- 5.4 (*) A *self-polar* triangle is a triangle polar to itself.
- (a) Show that a self-polar triangle does exist.
 (b) Show that all self-polar triangles are congruent.
- 5.5 On the planet Polaris the whole polar to each point of the dry land lies in the ocean.
- (a) How many continents may be on the Polaris if every continent is a disc?
 Here by a disc centred at p_0 of radius r we mean the set $\{p \in S^2 \mid d(p, p_0) < r\}$.
 Is the number of continent bounded? Can it be odd?
- (b) Is it possible that the whole polar to each point of the ocean belongs to the dry land?
- 5.6 Prove the formulae for a spherical triangle with right angle γ :
- (a) $\tan a = \tan \alpha \sin b$ (b) $\tan a = \tan c \cos \beta$.
- 5.7 Let T be a spherical triangle with three right angles. Let r and R be the radii of the inscribed and superscribed circles for T . Find the ratio $\sin R / \sin r$.
- 5.8 (*) For a spherical triangle with angles $\frac{\pi}{2}, \frac{\pi}{4}, \frac{2\pi}{3}$ on the unit sphere find the length of the side opposite to the angle $\frac{2\pi}{3}$.
- 5.9 (a) Given a spherical line segment of length α , prove that the polars of all spherical lines intersecting this segment sweep out a set of area 4α .
 (b) Given several spherical line segments whose sum of lengths is less than π , prove that there exists a spherical line disjoint from each.
- 6.1 (a) Find the area of a spherical triangle with angles $\frac{\pi}{2}, \frac{\pi}{3}$ and $\frac{\pi}{3}$. Which part of the area of the whole sphere does it make?
 (b) The same question for the triangle with angles $\frac{\pi}{2}, \frac{\pi}{3}$ and $\frac{\pi}{4}$.
- 6.2 (a) Find the area of a spherical quadrilateral with angles $\alpha, \beta, \gamma, \delta$.
 (b) Given the angles of a spherical n -gone, find its area.
- 6.3 Let $Ant : S^2 \rightarrow S^2$ be the antipodal map (which takes every point of the sphere to its antipodal). Write Ant as a composition of reflections.
- 6.4 Show that the group $Isom^+(S^2)$ of orientation preserving isometries of the sphere is generated by rotations by angle π .
- 6.5 (*) Prove that (a) the medians and (b) the altitudes of a spherical triangle are concurrent.
Remark: for part (b) assume that the triangle has at most one right angle.
Hint: use some projection to reduce the question to the similar questions on \mathbb{E}^2 .
- 6.6 Given a spherical triangle ABC and the midpoints M and N on the sides AB and AC respectively, show that $MN > BC/2$.

References:

1. Lectures 9-11 follow Prasolov's book (see Lecture I and pp. 48-49).
You can find the same material in pp. 83-87 of Prasolov, Tikhomirov.
2. Lecture 12 does not follow any particular text. The spirit follows the paper by Oleg Viro:
 - O. Viro, *Defining relations for reflections. I*, arXiv:1405.1460v1.
3. Another discussion of the isometry group of the sphere may be found in
 - G. Jones, *Algebra and Geometry*, Lecture notes (Section 2.2).