

Questions for Problems classes

Here are some questions which will be probably discussed in the Problems Class (subject to change!)

1 Reflections on the plane, Geometric constructions

Problems Class 1 (17 October, 2023)

1. Let $R_{A,\varphi}$ and $R_{B,\psi}$ be rotations with $0 < \varphi, \psi \leq \pi/2$. Find the type of the composition $f = R_{B,\psi} \circ R_{A,\varphi}$.
Hint: This is an example of using reflections to study compositions of isometries (write everything as a composition of reflections, make your choice so that some of them cancel!).
2. Let A and B be two given points in one half-plane with respect to a line l . How to find a shortest path, which starts at A then travels to l and returns to B ? (How to find the point where this path will reach the line l ?)
3. Ruler and compass constructions: perpendicular bisector, perpendicular from a point to a line, midpoint of a segment, angle bisector, inscribed and circumscribed circles for a triangle.

2 Group actions on \mathbb{E}^2

Problems Class 2 (31 October, 2023)

0. Let g_1, \dots, g_n be isometries of \mathbb{E}^2 . Let $G = \langle g_1, \dots, g_n \rangle$ be the group generated by g_1, \dots, g_n (i.e. the minimal group containing all of g_1, \dots, g_n). Show that the group G acts on \mathbb{E}^2 .
1. Let G be a group generated by two reflections on \mathbb{E}^2 . When G is discrete?
2. Let T be a triangle with angles $\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}$. Let r_1, r_2, r_3 be the reflection with respect to the sides of T , and let G be the group generated by r_1, r_2, r_3 . In the lecture we have checked that $G : \mathbb{E}^2$ discretely. Find the fundamental domain of this action.
3. Find the orbit-space for the action introduced in Question 2.
4. Let X be a regular triangle on \mathbb{E}^2 . Let r_1 and r_2 be two distinct reflections taking X to itself. Find the fundamental domain of the action $G : X$. Find also the orbit-space.
5. Let G be a group generated by rotation through angle $\frac{2\pi}{3}$ on the plane. Find the orbit-space of the action $G : \mathbb{E}^2$. Are there closed geodesics in this orbit-space? Are there bi-infinite open geodesics?

3 Spherical Geometry

Problems Class 3 (14 November, 2023)

1. Let $G : S^2$ be an action. G acts discretely if and only if $|G| < \infty$.
2. Let $G : X$ be an action and suppose that F is its fundamental domain. Then one can show that the action $G : X$ is discrete.
3. Let g be a reflection, $h \in Isom(S^2)$. h is a reflection if and only if there exists $f \in Isom(S^2)$ such that $fgf^{-1} = h$.
4. Let S^2 be a sphere of radius 1. Show that the length of a circle of (spherical) radius r equals to $2\pi \sin r$.
Remark: for the sphere of radius R , the length of the circle of radius r will be $2\pi R \sin(\frac{r}{R})$. When $R \rightarrow \infty$ we see that $\frac{r}{R} \rightarrow 0$ and, hence, $2\pi R \sin(\frac{r}{R}) \rightarrow 2\pi r$.
5. Let S^2 be a sphere of radius R . Let α and β be two parallel planes crossing S^2 . Find the area of the part of S^2 lying between the planes α and β .
6. One can also discuss ruler and compass constructions, as in \mathbb{E}^2 .

4 Projective geometry

Problems Class 4 (28 November, 2023)

1. Find a projective transformation f which takes

$$\begin{aligned} A &= (1 : 0 : 0) \text{ to } (0 : 0 : 1) \\ B &= (0 : 1 : 0) \text{ to } (0 : 1 : 1) \\ C &= (0 : 0 : 1) \text{ to } (1 : 0 : 1) \\ D &= (1 : 1 : 1) \text{ to } (1 : 1 : 1) \end{aligned}$$

Find the image of $X = AD \cap BC$ under this transformation.

2. Find $[A, B, C, D]$ for the points above. (Does it exist?)
For $E = (1 : 1 : 0)$, $F = (1 : 2 : 0)$ find $[A, B, E, F]$.
3. Check explicitly, that the transformation f from Question 1 preserves the value of $[A, B, E, F]$.
4. Let A_1, A_2, A_3, A_4 be points on a line a , let B_1, B_2, B_3, B_4 be points on a line b . Denote by p_i the line through A_i and B_i . Show that if the lines p_1, p_2, p_3, p_4 are concurrent, then the points $A_{i+1}B_i \cap A_iB_{i+1}$ ($i = 1, 2, 3$) are collinear.
5. Formulate and prove the statement dual to the one in Question 4.

5 Möbius geometry

Problems Class 5 (30 January, 2025)

1. Find the type of Möbius transformation $f(z) = 1/z$.
2. Let f and g be inversions with respect to two intersecting circles. Show that $g \circ f = f \circ g$ if and only if $Fix_f \perp Fix_g$.
3. Let $\mathcal{C}_1, \dots, \mathcal{C}_5$ be circles all passing through the points A and B on the plane. Show that there exists a circle γ orthogonal to all of \mathcal{C}_i .
4. Prove Ptolemy's theorem: given a quadrilateral $ABCD$ inscribed into a circle, one has

$$|AB| \cdot |CD| + |BC| \cdot |AD| = |AC| \cdot |BD|.$$

6 Poincaré disc

Problems Class 6 (13 February, 2025)

1. Let l_1 and l_2 be two divergent lines. Show that there exists a unique common perpendicular to l_1 and l_2 .
2. Let $0 \leq \alpha, \beta, \gamma < \pi$, $\alpha + \beta + \gamma < \pi$. Then there exists a hyperbolic triangle with angles α, β, γ .
3. Show that every triangle in \mathbb{H}^2 has an inscribed circle.
4. Given a hyperbolic triangle, is it true that it always has a circumscribed circle?
5. Ruler and compass constructions in hyperbolic plane:
 - midpoint of a segment;
 - perpendicular bisector;
 - angle bisector;
 - centre of a given circle;
 - tangent line to a given circle;
 - centre of inscribed circle for a triangle;
 - centre of circumscribed circle for a triangle (when exists);
 - common perpendicular for two divergent lines (in assumptions that one is given an infinitely long ruler, which allows to draw lines through two points of the absolute).
6. A polygon with all vertices at the absolute is called *ideal*. Is it true that all ideal hyperbolic quadrilaterals are isometric to each other?

7 Some computations in hyperbolic plane

Problems Class 7 (27 February, 2024)

1. Show that area of hyperbolic disc of radius r is $4\pi \sinh^2(\frac{r}{2})$.
2. [Hyperbolic oranges] Consider hyperbolic oranges, i.e. a discs of radius r , where the inner disc of radius $\frac{9}{10}r$ is a pulp, while the outer $1/10$ is a peel. Assuming that all oranges are equally tasty, which oranges are better to buy: big or small?
3. [Hyperbolic golf] Assume you are playing golf on hyperbolic field and your aim is 300m apart from you. You made a very good shoot, by sending a ball 300m far away with only 1° mistake. How far away from you goal will the ball land?
4. We had many statements about hyperbolic triangles (e.g. sine and cosine rules, congruence theorems, area formula) - which of them still hold for unbounded triangles, having at least one vertex at the absolute?

8 Computations in hyperboloid model

Problems Class 8 (13 March, 2025)

1. Given a right-angled triangle with sides a, b, c and corresponding angles α, β, γ , where $\gamma = \pi/2$, show that
$$\sinh a = \sinh c \sin \alpha.$$
2. Prove Pythagorean theorem $\cosh c = \cosh a \cosh b$ for a right-angled triangle with $\gamma = \pi/2$.
3. For a right-angled triangle with $\gamma = \pi/2$ prove $\tanh b = \tanh c \cos \alpha$.
4. Let $ABCD$ be a hyperbolic quadrilateral, with $\angle A = \angle B = \angle C = \pi/2$, $AB = a$ and $BC = b$. Denote $\angle D = \varphi$. Find $\cos \varphi$.
5. Find the radius of the circle inscribed into an ideal hyperbolic triangle.
6. Let l be a line in \mathbb{H}^2 , e be an equidistant curve to the line. Let $N, N' \in l$ and $A \in e$. Consider the points $B = e \cap AN$ and $C = e \cap AN'$. Show that the area S_{ABC} of triangle $\triangle ABC$ does not depend on the choice of $A \in e$.

9 Revision

Revision will not be based on demonstrating solutions, but still, there will be a couple of illustrating examples in each lecture, so you can try solving them in advance.

Revision 1 (29 April, 2025)

1. Let $ABCD$ be a trapezoid on Euclidean plane. Let M and N be the midpoints of the parallel sides BC and DA respectively. Let $P = AB \cap CD$ and $Q = AC \cap BD$. Show that the points M, N, P, Q are collinear.
2. Let \mathcal{C}_1 and \mathcal{C}_2 be two intersecting circles. Suppose they are not tangent. Let \mathcal{C}_3 be a circle orthogonal to both \mathcal{C}_1 and \mathcal{C}_2 . And let \mathcal{C}_4 and \mathcal{C}_5 be circles tangent to all of $\mathcal{C}_1, \mathcal{C}_2$ and \mathcal{C}_3 (touching the same segment of \mathcal{C}_3). Show that $\mathcal{C}_4 \cap \mathcal{C}_5 \neq \emptyset$.

Revision 2 (2 May, 2025)

1. Let γ be a circle in \mathbb{H}^2 . Suppose that $f(\gamma) = \gamma$ for some isometry $f \in Isom(\mathbb{H}^2)$. Describe f .
2. Let $G = \langle g_1, g_2 \rangle$, where $g_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $g_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ act on \mathbb{R}^2 as linear transformations. Let $A = (0, 0), B = (1, 0), C = (0, 1)$.
 - (a) Is there an element $g \in G$ such that $g(\triangle ABC) = \triangle ACB$?
 - (b) Is there an element $g \in G$ such that $g(\triangle ABC) = \triangle ACD$, where $D = (0, 3)$?
 - (c) Is there an element $g \in G$ such that $g(\triangle ABC) = \triangle AFE$, where $E = (1, 1), F = (2, 1)$?
3. Let R be an anticlockwise rotation around the origin by angle $\pi/2$ on Euclidean plane. Let T be a translation by 1 (given by $z \rightarrow z + 1$).
 - (a) Describe the composition $f = T \circ R$.
 - (b) Does the group $G = \langle R, T \rangle$ act discretely on \mathbb{E}^2 ?