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Geometry, 2025/2026

Problems Classes

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22 October 2025

Question 1.1. Is the following statement true of false?

"The isometries of $\mathbb{E}^2$ taking $(0,0)$ to $(0,0)$ and $(0,1)$ to $(0,2)$ form a group"

<u>Solution:</u> A map taking $(0,0)$ to $(0,0)$ and $(0,1)$ to $(0,2)$ is not an isometry. So, the set of such maps is empty. The empty set contains no identity element - which means it cannot be a group.

<u>Answer:</u> NO.

Question 1.2. Let $R_{A,\varphi}$ and $R_{B,\psi}$ be rotations with $0 < \varphi, \psi \leq \pi/2$. Find the type of the composition $f = R_{B,\psi} \circ R_{A,\varphi}$.

<u>Solution:</u> This is an example of using reflections to study compositions of isometries (we will write everything as a composition of reflections, making our choices so that some of them will cancel).

Notice that f preserve the orientation. Hence, it is either identity map, or rotation or translation. Furthermore, uniqueness part of Theorem 1.10 implies that $f = id$ if and only if $R_{A,\varphi} = R_{B,\psi}^{-1}$. In other words, $f = id$ if and only if $A = B$ and $\varphi = -\psi$.

To determine when f is a rotation and when it is a translation we write each of $R_{A,\varphi}$ and $R_{B,\psi}$ as a composition of two rotations (so that $f = r_4 \circ r_3 \circ r_2 \circ r_1$). Let l be the line through A and B. Then there exist lines l' and l'' such that $R_{B,\psi} = r_{l''} \circ r_l$ and $R_{A,\varphi} = r_l \circ r_{l'}$. Hence,

$$f = r_{l''} \circ r_l \circ r_l \circ r_{l'} == r_{l''} \circ r_{l'}.$$

Therefore f is a translation if $l' || l''$ and a rotation otherwise.

Finally, since $R_{B,\psi} = r_{l''} \circ r_l$, the angle from l to l'' is $\psi/2$. Also, since $R_{A,\varphi} = r_l \circ r_{l'}$, the angle from l' to l equals $\varphi/2$, see Fig.1. Since $0 < \varphi, \psi < \pi/2$, we see that f is always a rotation.

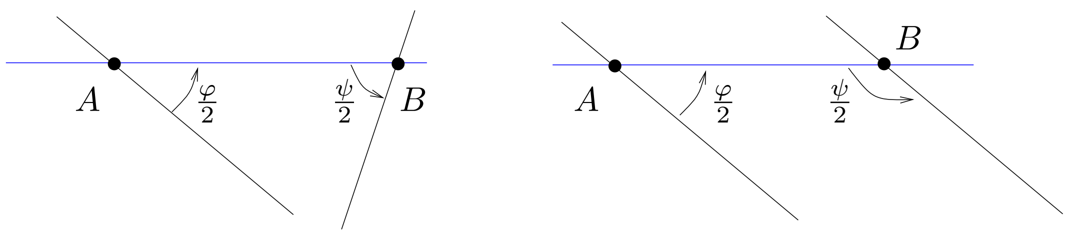

Figure 1: Question 1.1.

Question 1.3. Let A and B be two given points in one half-plane with respect to a line l. How to find a shortest path, which starts at A then travels to l and returns to B? (How to find the point where this path will reach the line l?)

<u>*Solution:*</u> Consider the point B' symmetric to B with respect to the line l. Then the shortest path from A to B' is the segment AB'. Let $M = AB' \cap l$, see Fig. 2. We claim that the broken line AMB (travelling from A to M and then to B) is the shortest path from A to B visiting a point on l.

Indeed, for any path γ from A to B visiting a point $Q \in l$ there exists a path γ' from A to Q and then from Q to B' such that the length of γ is the same as the length of γ' (we just reflect the part QB with respect to l). Since AB' is the shortest path from A to B', the broken line AMB is shorter than any other path from A to B vising the line l.

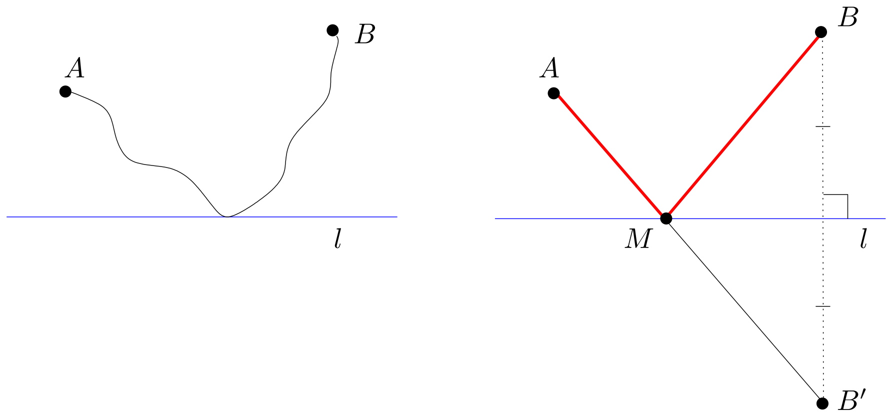

Figure 2: Question 1.2.

Question 1.4 (Geometric constructions)**.** By geometric constructions we mean constructions with ruler and compass. Here, a <u>ruler</u> is an instrument allowing to draw a line AB through two given points A and B. And a <u>compass</u> is an instrument allowing to draw a circle $C_A(AB)$ with the centre A and radius $\overline{AB}$. In this question we discuss how to construct the following sets:

(a) perpendicular bisector,

(b) midpoint of a segment,

(c) perpendicular from a point to a line,

(d) angle bisector,

(e) circumscribed circle for a triangle,

(f) inscribed circle for a triangle.

Solution:

(a) **Perpendicular bisector.** Given a segment AB, we need to construct a line l such that $l \perp AB$ and the point $M = l \cap AB$ is a midpoint for AB (i.e. $AM = MB$).

Construction: Let A and B be two points. To construct their perpendicular bisector, consider the circles $C_A(AB)$ and $C_B(AB)$ of radius AB centred at A and B respectively. Let X and Y be the two points of intersection of these two circles. (Their existence is due to continuity axiom - or we can obtain the point by a computation on $\mathbb{R}^2$). Then the line l_{XY} through the points X and Y is the perpendicular bisector for AB.

Proof: Let $M = XY \cap AB$. We need to show that $AM = MB$ and $\angle AMX = \angle BMX$. Notice that $\triangle AXY \cong \triangle BXY$ (by SSS), and hence, $\angle AXY = \angle BXY$. Furthermore, $\triangle AXM \cong \triangle BXM$ (by SAS), and hence $AM = BM$ and $\angle AMX = \angle BMX$.

Remark. Notice that we just proved that the locus of points on the same distance from A and B is the perpendicular bisector (E14).

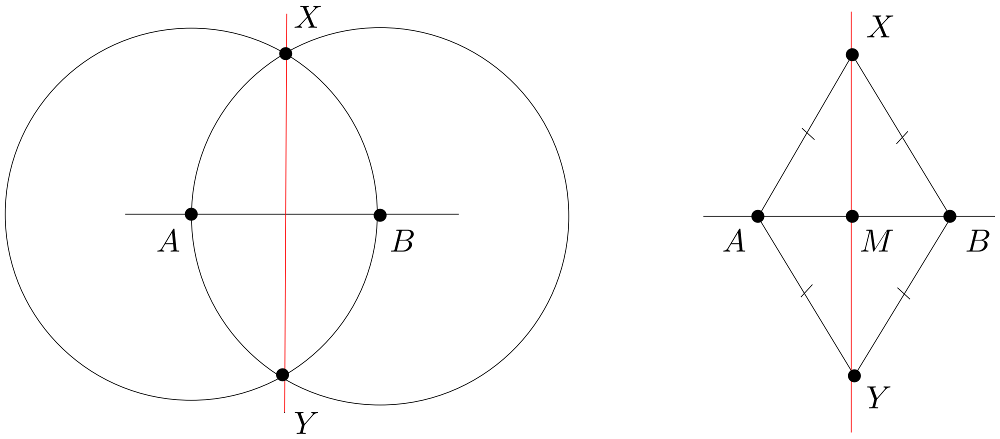

Figure 3: Question 1.3 (a): Construction of perpendicular bisector

Remark. (Extracted from the chat during the problems class).
One can do the same construction with circles centred at A and B of any equal radii - I do not need to require this radius to be AB. Then the same proof (which did not use that $AX = AB$!) will show that the construction still works. As the proof only uses that the points X, Y lie on the same (and now random!) distance from A and B, this proves in addition that the locus of points on the same distance from A and B coincides with the perpendicular bisector!

(b) **Midpoint for a segment.** This immediately follows from the construction (a).

(c) **Perpendicular from a point to a line.** Given a line l and a point $A \notin l$ we need to construct a line l' such that $A \in l'$ and $l' \perp l$.

Construction: Let $C_A(r)$ be a circle centred at A with radius $r > d(A, l)$ (where $d(A, l)$ denotes distance from A to the closest point of l). Consider the points X and Y where t $C_A(r)$ intersects l (they do exist as r is big enough). Let l' be the perpendicular bisector to XY. We claim that $A \in l'$ and $l' \perp l$.

Proof: Since $l = XY$ and l' is perpendicular to XY we have $l' \perp l$. So, we only need to prove that $A \in l'$. We know that $AX = AY$ and that the perpendicular bisector is the locus of points on the same distance from X and Y (E14), so, we conclude that $A \in l'$.

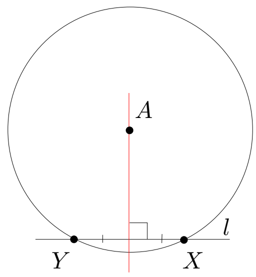

Figure 4: Question 1.3 (c): Construction of a perpendicular from a point to a line

(d) **Angle bisector.** Given an angle $\angle BAC$, we need to construct a ray AM such that $\angle BAM = \angle MAC$.

Construction: Let $C_a(r)$ be a circle centred at A of any radius r. Let $X = C_a(r) \cap AB$ and $Y = C_a(r) \cap AC$. Let l be the perpendicular bisector for the segment XY. Then l is the angle bisector for $\angle BAC$.

Proof: Notice that since $AX = AY = r$, we conclude that $A \in l$ (as the perpendicular bisector for XY is the locus of points on the same distance from X and Y by E14). Now, let $M = XY \cap l$. Then $\triangle AXM \cong \triangle YAM$ by SSS, which implies that $\angle XAM = \angle YAM$.

Remark-Exercise. An angle bisector is a locus of points on the same distance from the sides of the angle.

Hint: Given a point N on the angle bisector (resp. on the same distance from the sides of the angles), drop perpendiculars NX' and NY' on the sides of the angle and notice that $\triangle ANX' \cong \triangle ANY'$ (why?). Conclude from this that N lies on the same distance from the sides of the angle (resp. lies on the angle bisector).

(e) **Circumscribed circle for a triangle.** Given three non-collinear points A, B, C, we need to construct a circle through these points.

Construction: Let l_A be the perpendicular bisector for BC and l_B be the perpendicular bisector for AC. Then $O = l_A \cap l_B$ is the centre of the required circle.

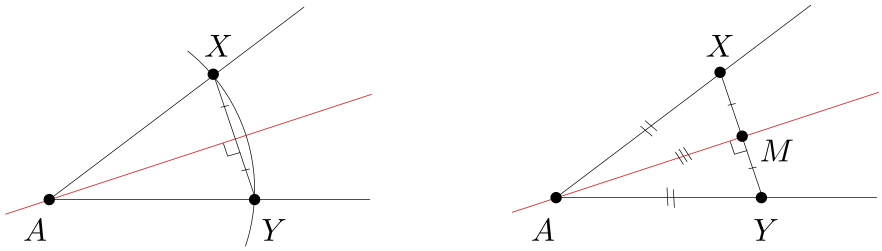

Figure 5: Question 1.3 (d): Construction of an angle bisector

Proof: We need to show that $OA = OB = OC$. Note that $OB = OC$ since $O \in l_A$, also $OA = OC$ since $O \in l_B$. This implies the statement.

Corollary. The three perpendicular bisectors in a triangle are concurrent.

Proof: As $OA = OB$, i.e. O lies on the same distance from A and B, we conclude (again by E14) that O lies on the perpendicular bisector for AB. So, the three perpendicular bisectors are concurrent at O.

(f) **Inscribed circle for a triangle.** Given a triangle ABC, we need to construct a circle which is tangent to all three sides of ABC.

Construction: Let l_A be the angle bisector for $\angle A$ and l_B be the angle bisector for $\angle B$. Then $O = l_A \cap l_B$ is the centre of the required circle. To find the radius we drop a perpendicular from O to one of the sides.

Proof: We need to show that O lies on the same distance from the lines AB, AC and BC. As $O \in l_A$, we know that O lies on the same distance from AB and AC (see the remark above!), and as $O \in l_B$, we see that O lies on the same distance from AB and CB. We conclude that O lies on the same distance from all three sides (so, if r is that distance then the circle $C_O(r)$ is tangent to all three sides and hence is the inscribed circle for $\triangle ABC$).

Corollary. The three angle bisectors in a triangle are concurrent.

Proof: As O lies on the same distance from all three sides, we conclude that it also lies on the angle bisector l_C for angle $\angle BCA$. So, three angle bisectors are concurrent at the point O.

Remarks:

- A solution for a construction question should always contain two parts:
(i) construction (i.e. the algorithm for the construction) and
(ii) justification (i.e. the proof that the construction provides the required object).

- One does not really need to have a ruler and a compass to solve questions on ruler and compass constructions. Moreover, I think that using the real instruments and drawing ideal diagrams does not really help to solve the questions but just distracts.

Remark on constructability. Not everything is contractible with ruler and compass. Here are several classical examples.

- **Squaring a circle:** given a circle, construct a square of the same area as the circle. This is equivalent to constructing a segment of the length $\sqrt{\pi}$ given a segment of the length 1.

- **Duplicating a cube:** given a cube of volume V construct the cube of volume $2V$. This is equivalent to constructing a segment of length $2^{1/3}$ given a segment of length 1.

- **Trisecting an angle:** Given an angle θ, construct and angle of size $\theta/3$.

For explanations why these constructions are impossible one can use field extensions, see

- Gareth Jones, *Algebra and Geometry*, Section 8.

(You will be able to find the notes by Gareth Jones on Ultra, in the Other Resources section).

We have used ruler and compass, but one can prove that everything constructible with ruler and compass can be constructed with compass alone. This statement is called Mohr-Mascheroni Theorem. See the following paper for a short and elementary proof:

- N. Hungerbuhler *A Short Elementary Proof of the Mohr-Mascheroni Theorem*, The American Mathematical Monthly, Vol. 101, No. 8 (Oct., 1994), pp. 784-787 (4 pages), https://www.jstor.org/stable/2974536.