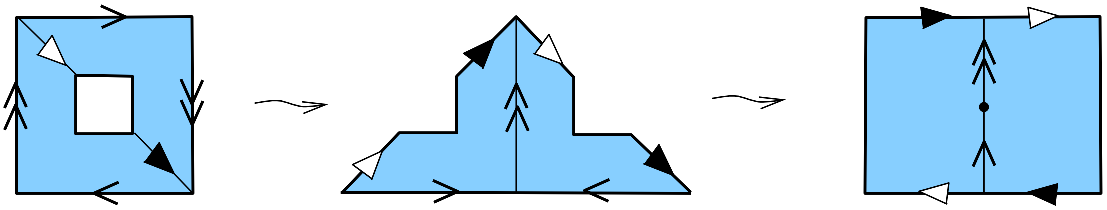
Solutions 9-10

9.1 Show that removing a small disc from a projective plane we get a Möbius band.

Solution. One can use the following "cut and paste" argument:



9.2 Removing a line from $\mathbb{R}^2$ or from S^2 one gets a space with two connected components.
Show that $RP^2 \setminus \mathbb{R}P^1$ is a connected space.

Solution. Recall that $\mathbb{R}P^2$ may be obtained obtained from S^2 by identification of antipodal points. Let as remove the line *before* the identification: we will get two open hemispheres. After the identification we get an open disc.

9.3. Let the Möbius band $\mathbf{M}$ is obtained by gluing along the vertical sides of the square with vertices $(\pm 1, \pm 1)$. Let m be a midline of the Möbius band $\mathbf{M}$ (obtained from the segment of the line line $y = 0$).

(a) what is $\mathbf{M} \setminus m$?

(b) Let l be the closed line obtained from $y = 1/2$ and $y = -1/2$. What is $\mathbf{M} \setminus l$?

Solution. Glue the Möbius band from a piece of paper, take the scissors and cut along m or l respectively.

9.4. Let $\mathbf{C}$ be the conic $x^2 + y^2 = z^2$. What kind of space is $\mathbb{R}P^2 \setminus \mathbf{C}$?

Solution. Looking at a hemisphere it is easy to see that $\mathbb{R}P^2 \setminus \mathbf{C}$ has two connected component, one is the disc ("inside" of the conic), the other is a projective plane without disc, which is a Möbius band as shown in Question 9.1.

9.5. Given a point P inside a circle and a chord AB through the point P, let M_{AB} denote the intersection point of the two lines tangent to the circle at A and B. Show that M_{AB} runs over some line as A runs over the circle.

Solution. When A runs over a circle, M_{AB} runs over all points polar to A, so we get a line polar to A.

9.6. Let $\mathbf{C}$ be the conic $x^2 + y^2 = z^2$. A triangle in $\mathbb{R}P^2$ is self-polar (with respect to $\mathbf{C}$) if its sides are polar to its vertices (not necessarily the opposite ones).

(a) Construct a self-polar triangle with two vertices on $\mathbf{C}$.

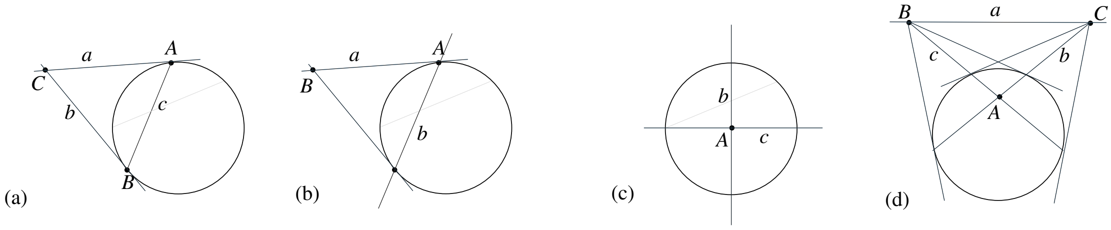
(b) Does there exist a self-polar triangle with exactly one vertex on $\mathbf{C}$?

(c) Show that there exists a self-polar triangle having no vertex on $\mathbf{C}$.
Hint: it may have some vertices at infinity.

Solution. (a) A point $A \in \mathbf{C}$ is polar to a line l_A tangent to $\mathbf{C}$ at A. So, if we take a point C outside the conic, and consider the tangent lines L_A and L_B through C (tangent to $\mathbf{C}$ at A and B respectively), then $c = AB$ is polar to C, A is polar to a and B is polar to b.

(b) Suppose that such a configuration ABC does exist. Let A be the vertex on $\mathbf{C}$ and a the tangent at A. The side a of the triangle should have two vertices on it, one is A let B be the other one. Next, let b be the side of the triangle polar to B: it should pass through A and another point where a line through B is tangent to $\mathbf{C}$. The last vertex C of the triangle ABC should lie on the intersection of b and the line c polar to C. By assumption, $C \notin \mathbf{C}$. If C lies inside the conic then the polar line c lies completely outside the conic, so $C \notin c$. If C lies outside the conic, then c intersects b inside the conic, so $C \neq b \cap c$.

(c) One can take a triangle with A at the centre of the circle, and B and C at infinity, so that the lines b and c form a right angle and line a is the line at infinity.



Remark. If you want to see such a configuration without vertices at infinity, look at Figure (d). It rather easy to see that this configuration does exist (from continuity reasons), and more difficult (but still possible) to find an explicit construction.

9.7. (a) Formulate the theorem dual to Desargues' theorem.

(b) Draw an example. (Hint: send the line s to the line at infinity).

(c) Can you prove this theorem?

Solution. (a) We need to substitute: points by lines, lines by points,
a line through two points by a point of intersection of two lines,
intersection of two lines by a line through two points,
concurrency of lines by collinearity of points (and backwards).

The assumption was:

> *Suppose that the lines joining the corresponding vertices of triangles $A_1A_2A_3$ and $B_1B_2B_3$ intersect at one point S.*

Denote by a_i and b_i the sides of the triangles opposite to A_i and B_i respectively. Then the assumption says:

> *Suppose that the lines A_1B_1, A_2B_2, A_3B_3 intersect at one point S.*

We rewrite:

> *Suppose that the points $a_1 \cap b_1$, $a_2 \cap b_2$, $a_3 \cap b_3$ lie on one line s.*

The conclusion was

> *Then the points $P_1 = A_2A_3 \cap B_2B_3$, $P_2 = A_1A_3 \cap B_1B_3$, $P_3 = A_1A_2 \cap B_1B_2$ are collinear.*

Instead of points P_i we will get lines p_i:
p_1 is a line through $a_2 \cap a_3$ and $b_2 \cap b_3$, in other words, $p_1 = A_1B_1$, and similarly, $p_2 = A_2B_2$, $p_3 = A_3B_3$.

So, we get the following conclusion:

> *Then the lines $p_1 = A_1B_1$, $p_2 = A_2B_2$, $p_3 = A_3B_3$. are concurrent.*

So, the dual to the Desargues' Theorem is just the converse statement:

> *Suppose that the points $a_1 \cap b_1$, $a_2 \cap b_2$, $a_3 \cap b_3$ lie on one line s.*
> *Then the lines $p_1 = A_1B_1$, $p_2 = A_2B_2$, $p_3 = A_3B_3$. are concurrent.*

(b)... and we actually don't need to draw anything new!

(c) If we send the line s to infinity, then we get two triangles with corresponding sides parallel: a_i is parallel to b_i. Let $S = A_1B_1 \cap A_2B_2$. Then it is easy to see that there exists a homothety with centre S which takes points A_1 to B_1, A_2 to B_2. It also takes the line A_1A_3 to the line parallel to it and passing through A_1, i.e. to B_1B_3. Similarly, it takes the line A_2A_3 to B_2B_3. This implies that $A_3 = A_1A_3 \cap A_2A_3$ is mapped $B_3 = B_1B_3 \cap B_2B_3$. Hence B_3 lies on the line SA_3, i.e. $S \in A_3B_3$, and te lines A_1B_1, A_2B_2, A_3B_3 are concurrent at S.

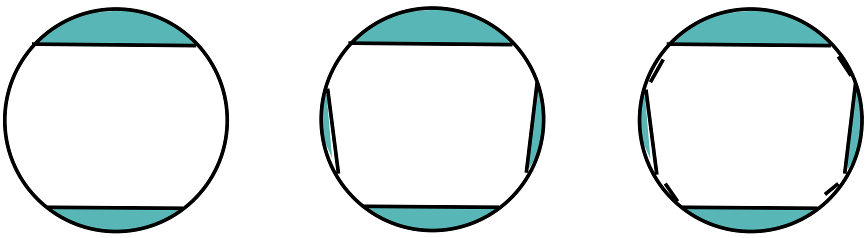
10.1. (a) How many non-intersecting lines can you draw in the Klein model of hyperbolic plane?
(b) The same question, but no other line should intersect more than two lines of your family.

Solution. (a) Choosing the segments of parallel (in Euclidean sense) lines we may get as as many mutually non-intersecting hyperbolic likes as we want (even uncountably many).

(b) To get countably many lines in the family we proceed as follow:

1) Take two disjoint lines
and color the two disjoint circular segments bounded by these lines;
2) In the uncolored domain insert a line between each two lines
and color the respective circular segment;
3) return to Step 2).

Here is the diagram for the first three iterations:

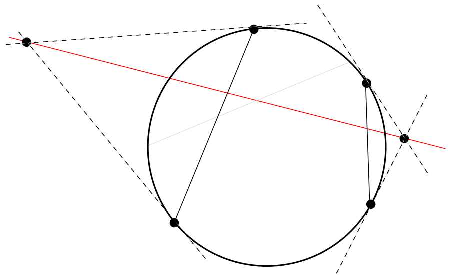


Now, suppose that a line l intersects two lines of the family. Then both ends of l should lies in the corresponding colored circular segments. It is clear that such a line will not intersect any other line in our family.

It is left to prove that it is impossible to find an uncountably big family of lines. First, suppose that in our family there are three lines $l_1, l_2 and l_3$ such that the line l_3 is contained in the union of the disjoint circular segments bounded by l_1 and l_2. Then it is easy to construct a line l intersecting all three lines l_1, l_2 and l_3. This means that in our family all lines come with the corresponding circular segment (and all these circular segments are mutually disjoint). As every circular segment contains a boundary point which makes a rational angle with horizontal line, we see that the number of the segments in our family is countable (here we use countability of rational numbers).

10.2. Show that any two lines on hyperbolic plane either intersect inside the hyperbolic plane, or intersect on its boundary, or have a unique common perpendicular.

Solution. Let AB and CD be two lines having no intersection neither inside nor on the boundary of the hyperbolic plane. Let l_A, l_B, l_C, and l_D be the lines tangent to the disc. A line orthogonal to AB in the Klein model should pass through the intersection point $X = l_A \cap l_B$. Similarly, the line orthogonal to CD should pass through $Y = l_C \cap l_D$. There exists a unique line on (Euclidean plane) through X and Y, it gives the common perpendicular. See on the diagram:



10.3. (Right-angled polygons on hyperbolic plane)

(a) Show that a hyperbolic triangle can not have more than 1 right angle.

(b) Show that there are no hyperbolic rectangles (i.e. quadrilaterals with 4 right angles).

(c) In the Klein model, construct a hyperbolic pentagon with 5 right angles.

Solution. (a) Let C be a right-angled vertex of the triangle, without loss of generality (i.e. by transitivity of isometry group on the points of the model) we may assume that C is the centre of the disc. The two sides a and b through C will be two orthogonal (in Euclidean sense) diameters. Now, a hyperbolic line orthogonal to a diameter (in hyperbolic sense) is also orthogonal to it in Euclidean sense. So if c is orthogonal to a then it will never meet b inside the model, and we can not get a triangle.

(b) Similarly to above, let a and b be two orthogonal sides meeting at the centre of the disc (say, a horizontal, b vertical). Then if $abcd$ is a rectangle, c should be horizontal and d vertical. But it is easy to see that in this case c and d does not form a right angle (d can not pass through the intersection point X of the tangents to the ends of c as X belongs to (continuation of) b and d is parallel to b).

(c) Using the same four lines as above (a horizontal diameter, b vertical diameter, c horizontal, d vertical) we can place c and d far enough from the centre so that $c \cap d = \emptyset$ in hyperbolic plane. Then by Question 10.2 there exists a common perpendicular to c and d (and we know how to construct it):