

**Assignment 13-14**  
**Starred problems due on Monday, 23 February**

13.1. Draw in each of the two conformal models (Poincaré disc and upper half-plane):

- (a) two intersecting lines;
- (b) two parallel lines;
- (c) two ultra-parallel lines;
- (d) infinitely many disjoint (hyperbolic) half-planes;
- (e) a circle tangent to a line.

13.2. In the upper half-plane model draw

- (a) a (hyperbolic) line through the points  $i$  and  $i + 2$ ;
- (b) a (hyp.) line through  $i + 1$  orthogonal to the (hyp.) line represented by the ray  $\{ki \mid k > 0\}$ ;
- (c) a (hyperbolic) circle centred at  $i$  (just sketch it, no formula needed!);
- (d) a triangle with all three vertices at the absolute (such a triangle is called *ideal*).

13.3. Prove SSS, ASA and SAS theorems of congruence for triangles on hyperbolic plane.

13.4. Let  $ABC$  be a triangle. Let  $B_1 \in AB$  and  $C_1 \in AC$  be two points such that  $\angle AB_1C_1 = \angle ABC$ . Show that  $\angle AC_1B_1 > \angle ACB$ .

13.5. Show that there is no “rectangle” in hyperbolic geometry (i.e. no quadrilateral has four right angles).

13.6. (\*) Given an acute-angled polygon  $P$  (i.e. a polygon with all angles smaller or equal to  $\pi/2$ ) and lines  $m$  and  $l$  containing two disjoint sides of  $P$ , show that  $l$  and  $m$  are ultra-parallel.

14.7. Given  $\alpha, \beta, \gamma$  such that  $\alpha + \beta + \gamma < \pi$ , show that there exists a hyperbolic triangle with angles  $\alpha, \beta, \gamma$ .

14.8. Show that there exists a hyperbolic pentagon with five right angles.

14.9. (\*) An *ideal* triangle is a hyperbolic triangle with all three vertices on the absolute.

- (a) Show that all ideal triangles are congruent.
- (b) Show that the altitudes of an ideal triangle are concurrent.
- (c) Show that an ideal triangle has an inscribed circle.

14.10. (\*) We know that an isometry fixing 3 points of the absolute is the identity map. How many isometries fix two points of the absolute? Classify the isometries fixing 0 and  $\infty$  in the upper half-plane model.

14.11. (a) Show that the group of isometries of hyperbolic plane is generated by reflections.  
 (b) How many reflections do you need to map a triangle  $ABC$  to a congruent triangle  $A'B'C'$ ?

14.12. (\*)

- (a) Does there exist a regular triangle with all vertices in the hyperbolic plane (not lying at the absolute)?
- (b) Does there exist a right-angled regular polygon in the hyperbolic plane? How many edges does it have (if exists)?

14.13. (a) Show that the angle bisectors in a hyperbolic triangle are concurrent.  
 (b) Show that every hyperbolic triangle has an inscribed circle.  
 (c) Does every hyperbolic triangle have a circumscribed circle?