Anna Felikson, Durham University **Geometry**, 6.2.2026

Assignment 15-16
Starred problems due on Friday, 9 March

15.1. (*)

(a) Let P and Q be feet of the altitudes in an ideal hyperbolic triangle. Find PQ.

(b) Find the radius of a circle inscribed into an ideal hyperbolic triangle.

(c) Show that a radius of a circle inscribed into a hyperbolic triangle does not exceed $arcosh(2/\sqrt{3})$.

15.2. For a right hyperbolic triangle ($\gamma = \frac{\pi}{2}$) show:

$$\text{(a)} \tanh b = \tanh c \cos \alpha, \qquad \text{(b)} \sinh a = \sinh c \sin \alpha.$$

15.3. Show that in the upper half-plane model the following distance formula holds:

$$2 \sinh^2 \frac{d}{2} = \frac{|z-w|^2}{2Im(z)Im(w)}$$

.

15.4. Find an area of a right-angled hyperbolic pentagon.

15.5. In the upper half-plane model, find the locus of points z lying on distance d from the line 0∞.

16.1. In the Klein disc model draw two parallel lines, two ultra-parallel lines, an ideal triangle, a triangle with angles $(0, \frac{\pi}{2}, \frac{\pi}{3})$.

16.2. (*) Show that three altitudes of a hyperbolic triangle either have a common point or are pairwise parallel or there is a unique line orthogonal to all three altitudes.

16.3. Let u, v be two vectors in $\mathbb{R}^{2,1}$. Denote $Q = |\frac{\langle u,v\rangle^2}{\langle u,u\rangle\langle v,v\rangle}|$, where $\langle x, y\rangle = x_1y_1 + x_2y_2 - x_3y_3$. Show the following distance formulae:

(a) if $\langle u, u\rangle < 0$, $\langle v, v\rangle < 0$, then u and v give two points in $\mathbb{H}^2$, and $\cosh^2(d(u, v)) = Q$.

(b) if $\langle u, u\rangle < 0$, $\langle v, v\rangle > 0$, then u gives a point and v give a line l_v on $\mathbb{H}^2$, and $\sinh^2 d(u, l_v) = Q$.

(c) if $\langle u, u\rangle > 0$, $\langle v, v\rangle > 0$ then u and v define two lines l_u and l_v on $\mathbb{H}^2$ and

- if $Q < 1$, then l_u intersects l_v forming the angle φ satisfying $Q = \cos^2 \varphi$;
- if $Q = 1$, then l_u is parallel to l_v;
- if $Q > 1$, then l_u and l_v are ultra-parallel lines satisfying $Q = \cosh^2 d(l_u, l_v)$.

16.4. (*) Consider the two-sheet hyperboloid model $\{u = (u_1, u_2, u_3) \in \mathbb{R}^{2,1} \mid \langle u, u\rangle = -1,\ u_3 > 0\}$, where $\langle u, u\rangle = u_1^2 + u_2^2 - u_3^2$.

(a) For the vectors

$$\begin{array}{lll} v_1 = (2,1,2) & v_2 = (0,1,2) & v_3 = (3,4,5) \\ v_4 = (1,0,0) & v_5 = (0,1,0) & v_6 = (1,1,2) \end{array}$$

decide if v_i corresponds to a point in $\mathbb{H}^2$, or a point in the absolute, or a line in $\mathbb{H}^2$.

(b) Find the distance between the two points of $\mathbb{H}^2$ described in (a).

(c) Which pair of the lines in (a) is intersecting? Which lines are parallel? Which are ultra-parallel?

(d) Find the distance between the pair of ultra-parallel lines in (a).

(e) Does any of the points in (a) lie on any of the three lines?

(f) Find the angle between the pair of intersecting lines.