Geometry III/V

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I devote this course to solidarity with Ukraine,
to solidarity with Israel,
with all peaceful people in hardship,
and with all people standing for peace and freedom!

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0 Introduction and History

0.1 Introduction

What to expect or 8 reasons to expect difficulties.

> *Our brain has two halves: one is responsible for multiplication of polynomials and languages, and the other half is responsible for orientation of figures in space and the things important in real life.*
> *Mathematics is geometry when you have to use both halves.*
>
> Vladimir Arnold

> *Geometry is an art of reasoning well from badly drawn diagrams.*
>
> Henri Poincaré

1. Structure of the course:

- It will be a zoo of different 2-dimensional geometries - including Euclidean, affine, projective, spherical and Möbius geometries, all of which will appear as some aspects of hyperbolic geometry.

  Why to study all of them?
  - They are beautiful!
  - we will need all of them to study hyperbolic geometry.

  Why to study hyperbolic geometry?
  - Important in topology and physics, for example.

  Example. When one looks at geometric structures on 2-dimensional closed surfaces, one can find out that only the sphere and torus carry spherical and Euclidean geometries on them, and infinitely many other surfaces (all other closed surfaces) are hyperbolic (see Fig. 1).
  (Given the time there will be more on that at the very end of the second term).

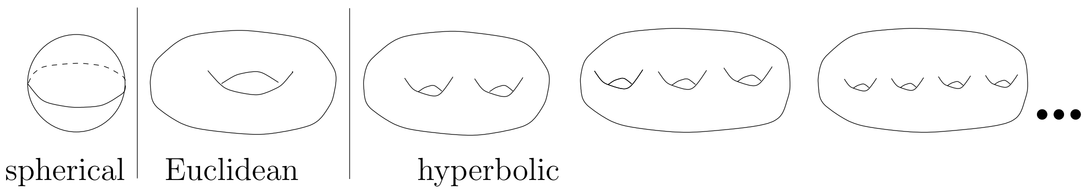

Figure 1: Geometric structures on surfaces.

- There will be just a bit on each geometry, hence the material may seem too easy.
- But it will get too difficult if you will miss something (as we are going to use extensively almost everything...)

2. Two ways of doing geometry: "synthetic" and "analytic"

- "Synthetic" way:
  - List axioms and definitions.
  - Then formally derive theorems.
  Question: is there any object satisfying the axioms?
  - Build a "model": an object satisfying the axioms (and hence, theorems).

- "Analytic" way:
  - Build a model
  - Work in the model to prove theorems (using properties of the model).

The same object may have many different models.

Example 0.1. A group $G_2 = \{e, r\} = \langle r | r^2 = e \rangle$
(Group with 2 elements, e, r, with one generator r and relation $r^2 = e$).
Model 1: Let r be a reflection on $\mathbb{R}^2$ (and e an identity map).
Model 2: $\{1, -1\} \in \mathbb{Z}$ with respect to multiplication.

We will sometimes use different models for the same geometry - to see different aspect of that geometry.

3. "Geometric" way of thinking:

Example 0.2. Claim. *Let ABC be a triangle, let M and N be the midpoints of AB and BC. Then $AC = 2MN$.*

We will prove the claim in two ways: geometrically and in coordinates. Geometric proof will be based on Theorem0.3.

Notation: given lines l and m, we write $l||m$ when l is parallel to m.

Theorem 0.3. *If ABC is a triangle, $M \in AB$, $N \in BC$, then $MN||AC \Leftrightarrow \frac{BA}{BM} = \frac{BC}{BN}$.*

Proof. (Geometric proof):
1) $MN||AC$ (by Theorem 0.3).
2) Draw $NK||AM$, $K \in AC$ (see Fig. 2, left).
3) Then $AK = KC$ (by Theorem 0.3).
4) The quadrilateral $AMNK$ is a parallelogram
(by definition of a parallelogram - as it has two pairs of parallel sides).
5) Hence, $MN = AK$ (by a property of a parallelogram).
6) $AC = 2AK = 2MN$ (by steps 3 and 5).

□

Proof. (Computation in coordinates):
We can assume that $A = (0, 0)$, $C = (x, 0)$, $B = (z, t)$ (see Fig. 2, right).
Then $M = (\frac{z}{2}, \frac{t}{2})$, $N = (\frac{x+z}{2}, \frac{z}{2})$.
Therefore, $MN^2 = (\frac{x+z}{2} - \frac{z}{2})^2 + (\frac{t}{2} - \frac{t}{2})^2 = (\frac{x}{2})^2$. Hence, $MN = x/2$, while $AC = x$.

□

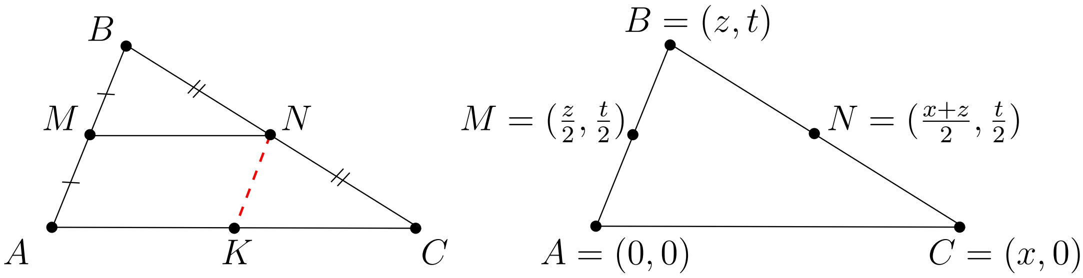

Figure 2: Two proofs of the theorem about midlines.

Note that even in the second proof we used geometry to simplify the computation: we assumed that $A = (0, 0)$, i.e. that all points of the plane are equally good, and that after taking A to the origin we can rotate the whole picture so that C get to the horizontal line.

4. We will use some results from Euclidean geometry without reproving.
- We need some basics.
- The complete way from axioms takes time.
- It is not difficult (was previously taught in schools).
- You can find proofs in books (will give some references).
- Hopefully, by now you have already mastered logical/mathematical thinking (and don't need the course on Euclidean geometry as a model for mastering them).

5. We will use many diagrams:
- They are useful
- but be careful: wrong diagrams may lead to mistakes.

Example 0.4. "Proof" that all triangles are isosceles (with demystification):
http://jdh.hamkins.org/all-triangles-are-isosceles/

6. Problem solving in Geometry
- Is not algorithmic (one needs practice!)
- Solution may be easy – but how to find it?
(additional constructions? which model to use? which coordinates to choose? ...
- all needs practice!)

For getting the practice we will have Problem Classes and Assignments:
- weekly sets of assignments;
- some questions will be starred - to submit for marking fortnightly (via Gradescope).
- other questions - to solve!
- There will be hints - use them if you absolutely don't know how to start the question without them (it is much better to attempt the questions with hints than just to read the solutions).

7. "Examples" will be hard to tell from "Theory":
"Problem"="one more theorem"
"Proof of a Theorem"="Example on problem solving".

8. Group approach to geometry

Klein's Erlangen Program: In 1872, Felix Klein proposed the following:
each geometry is a set with a transformation group acting on it.
To study geometry is the same as to study the properties preserved by the group.

Example 0.5. Isometries preserve distance;
Affine transformations preserve parallelism;
Projective transformations preserve collinearity;
Möbius transformations preserve property to lie on the same circle or line.

Why to speak about possible difficulties now?
- not with the aim to frighten you but
- to make sure you are aware of them;
- to inform you that they are in the nature of the subject;
- to inform you that I know about the difficulties- and will try my best to help;
- to motivate you to ask questions.

Remark. For seven top reasons to enjoy geometry check Chapter 0 here:
http://www1.maths.leeds.ac.uk/ kisilv/courses/math255.html