Riemannian Geometry

Hints 2

1. No extra hints...

2. We need to check that for each smooth function f the derivation $\frac{\partial}{\partial x_i}\big|_p$ acts in the in the same way as the derivation $\sum_{j=1}^{n} \frac{\partial y_j}{\partial x_i} \frac{\partial}{\partial y_j}\big|_p$. To do that we consider $f \circ \varphi^{-1} = f \circ \psi^{-1} \circ \psi \circ \varphi^{-1}$ and $f \circ \psi^{-1}$.

   Now, $f \circ \psi^{-1} \circ \psi \circ \varphi^{-1}$ is a composition of the two functions $\psi \circ \varphi^{-1} : V_1 \subset \mathbf{R}^n \to V_2 \subset \mathbf{R}^n$ and $f \circ \psi^{-1} : V_2 \subset \mathbf{R}^n \to \mathbf{R}$. This we can differentiate using the chain rule.

3. (*)

   (a) $\gamma'(0)(f) = \frac{d}{dt}\big|_{t=0} f(\gamma(t)) = ...$

   (b) Consider $\varphi \circ \gamma(t) : \mathbf{R} \to \mathbf{R}^2$, $t \to (\gamma_1(t), \gamma_2(t))$. Write $(\gamma_1(t), \gamma_2(t))$ explicitely and use the chain rule to find $\gamma'(t)$.

4. (a) Consider the sphere S^3 as a preimage of the regular value $F^{(-1)}(1)$ for the function $F(w, z) = f(a, b, c, d) = a^2 + b^2 + c^2 + d^2$. Compute $DF_{(1,0)}$.

   (b) Write the coordinates on $\mathbf{C}$ as $\alpha + i\beta$
   For the basis vectors $\frac{\partial}{\partial b}, \frac{\partial}{\partial c}, \frac{\partial}{\partial d}$ of $T_{(1,0)}S^3$ consider the curves γ_b, γ_c and γ_d such that the directional derivative along these curves coincide with $\frac{\partial}{\partial b}, \frac{\partial}{\partial c}, \frac{\partial}{\partial d}$. Then consider the image of these curves under the map π and write the directional derivatives along $\pi(\gamma_b)$, $\pi(\gamma_c)$ and $\pi(\gamma_d)$ in the basis $\langle \frac{\partial}{\partial \alpha}, \frac{\partial}{\partial \beta} \rangle$.