Riemannian Geometry

Hints 3-4

1. (a) Consider the function $F(A) = A^\top A - I$ and compute its differential using the chain rule.

   More precisely, you may want to use that the differential $Df(x)$ of a smooth function $f : \mathbf{R}^m \to \mathbf{R}^n$ satisfies $Df(u)(v) = \lim\limits_{t\to 0} \frac{f(u+tv)-f(u)}{t}$.

   (b) We need to check that the inverse element A^{-1} and the product AB are smooth functions (of elements of A and elements of A and B respectively) for all $A, B \in SO(n, \mathbf{R})$.

   (c) (*) For a curve $A(T)$ with $A(0) = I$ compute $DF(A)\mid_{t=0}$ (where $F(A) = A^\top A - I$).

2. To get good coordinates on $M \times N$ use the product atlas (see HW 1).

3. (a) Applying the definition $[X, Y]f = X(Y(f)) - Y(X(f)) = \dots$ you will get a long expression in partials. Since $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$, second derivatives will cancel and you will be left with $\varphi(x, y, z)\frac{\partial}{\partial x} + \psi(x, y, z)\frac{\partial}{\partial y} + \eta(x, y, z)\frac{\partial}{\partial z}$.

   (b) You need to check that the vectors X and Y at the point $p \in S^2$ are orthogonal to the vector pointing to the point p.

4. (*) A direct computation...

5. (a) Consider S^3 as a unit sphere inside $\mathbf{R}^4$. One example of the required vector field is $(-y, x, -w, z) \in T_{(x,y,z,w)}\mathbf{R}^4$.

   (b) Easy to obtain from (a).