Riemannian Geometry

Hints 5

1. Differentiate the equation $f(x_1, \dots, x_n) = a$.

2. (*)

   (a) – (c) explicit computation.

   (d) You need to compute $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}$. Then you will easily compute $\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle$. Then do the same for $\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}$ and compare (for $\langle \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_1} \rangle$ you will need to do a little bit more work to simplify the expression using basic property of hyperbolic functions).

3. To get the equality given in the hint remove the imaginary part from the denominator (i.e. multiply by the conjugate. Then find the differential $Df_A(z'(0))$ and use the hint to check that

$$\langle Df_A(z'(0)), Df_A(z'(0)) \rangle = \langle z'(0), z'(0) \rangle.$$

4. (a) Find the function $l : [a, b] \to [0, L(c)]$ then calculate its inverse and find the arc length reparametrization by $\gamma = c \circ l^{-1}$.

   (b) To compute $L(c)$ you need to integrate

$$||c'(t)||_{c(t)} = \sqrt{\frac{1}{\mathrm{Im}(c(t))} \langle c'(t), c'(t) \rangle} = \frac{1}{\mathrm{Im}(c(t))} |c'(t)|,$$

   so you need first to find $c'(t)$ and $\mathrm{Im}(c(t))$.

5. (a) Use that $d(iy_1, iy_2) = log(y_1/y_2)$ (shown in class).

   (b) Show that both LHS and RHS are preserved by isometries (for RHS you will need to use that $\mathrm{Im}(f_A(z)) = \mathrm{Im(z)}/|cz + d|^2$).

   (c) Draw a semicircle (or half-line) s orthogonal to the real axis and passing through z_1 and z_2. Show that there is an element of of $SL(2, \mathbf{R})$ which takes s to the upper half of the imaginary axis (to do that look at the intersection of s with the real axis: these points need to go to 0 and ∞).

   (d) Möbius transformations take circles and lines to circles and lines and also preserve angles.