Riemannian Geometry

Hints 7-8

1. (*) One may use the following plan:
   - write $X(t) = \sum a_i(t) \frac{\partial}{\partial x_i}$;
   - calculate Christoffel symbols;
   - use Γ_{ij}^k to find the action of the covariant derivative $\frac{D}{dt}$ on X;
   - write a system of ODEs using the parallel condition;
   - solve it
     (you may have sin and cos functions as coefficients, to get rid of them, specialize to their (constant) values on the curve c);
   - find X.
   - translate X into 3-dim expression.
2. Same plan as above (but without the last point).
3. (a) Compute the metric and use the formula $\Gamma_{ij}^s = \frac{1}{2} \sum_k g^{sk}(g_{jk,i} + g_{ik,j} - g_{ij,k})$.

   (b) Note that $\gamma_1'(t) = \frac{\partial}{\partial x_1}|_{\gamma_1(t)}$ and compute $\frac{D}{dt}\gamma_1'$ using Christoffel symbols.

   (c) Similarly, use $\gamma_2'(t) = \frac{\partial}{\partial x_2}|_{\gamma_2(t)}$
4. Direct computations.
5. (a) (*) Compute invariantly, using the Riemannian property.
   Remark: not every vector field along a curve may have a global extension; an extreme example is the case where $c : [a,b] \to M$ is a constant map $c(t) = p$ for all $t \in [a,b]$ and $X(t)$ is varying in T_pM.

   (b) Write X, Y with respect to a basis of a coordinate system and do the same computation as in (a) but in coordinates. For this, it is convenient to assume that $c([a,b])$ is contained in the domain of a coordinate chart. But this assumption is not a serious restriction, for otherwise one covers $c([a,b])$ with a finite sequence of covering coordinate charts and argues locally.