Riemannian Geometry, Michaelmas 2013.

Christmas Homework

Starred problems to be sent to Santa Claus by Tuesday, December 24th

Definition. Let (M, g) be a Riemannian manifold.
Let $\varphi : U \to V$ be a chart and $A \subseteq U \subseteq M$ be a subset.

A volume of A is defined as follows:

$$Vol(A) = \int_A d\ Vol = \int_{\varphi(A)} \sqrt{det(g_{ij})}dx,$$

where g_{ij} is the metric g written in the chart φ.

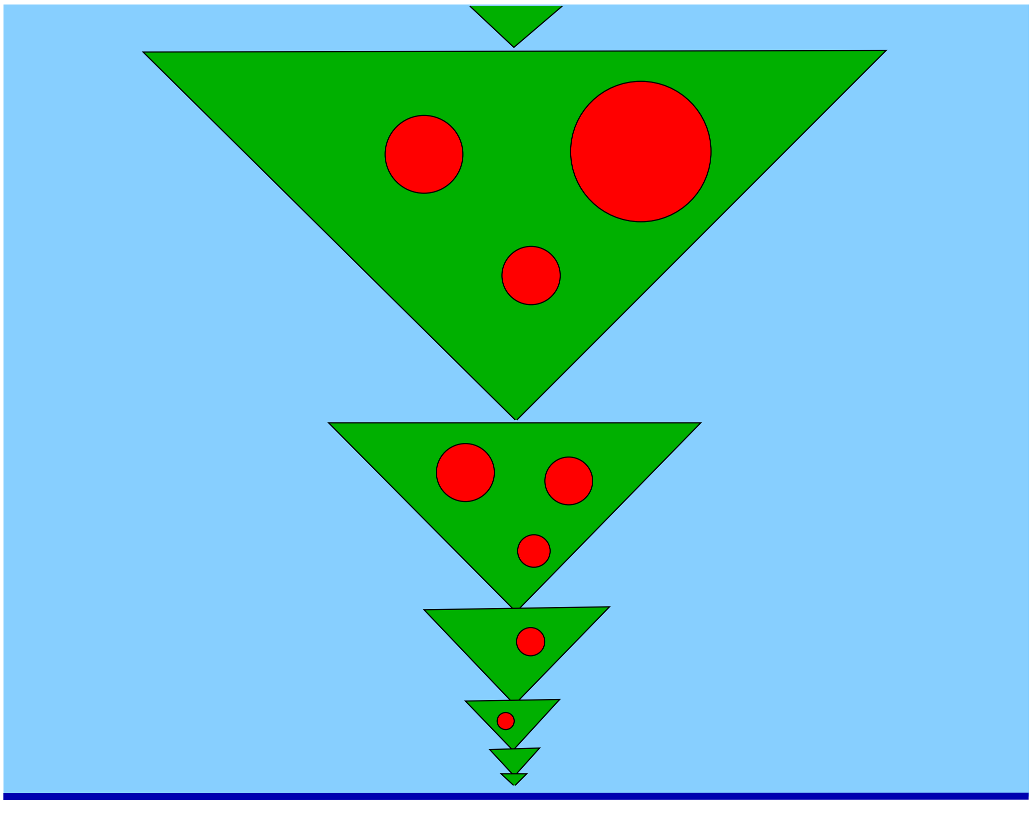
Example. Find the volume of the rectangle

$$A = \{z = x + iy \in \mathbf{C} \mid c \leq x \leq d, a \leq y \leq b\}$$

in the upper half-plane model of hyperbolic plane $(\mathbf{H}^2, g)$.

Solution. Recall, $\mathbf{H}^2 = \{z \in \mathbf{C} \mid y > 0\}$, and $\langle v, w\rangle_z = \frac{v_1w_1+v_2w_2}{y^2}$.
(we are working in one coordinate chart $\varphi(x, y) = (x, y)$).
Since $(g_{ij}) = \begin{pmatrix} 1/y^2 & 0 \\ 0 & 1/y^2 \end{pmatrix}$, we have $\sqrt{det(g_{ij})} = \frac{1}{y^2}$.
Hence, $Vol(A) = \int\limits_c^d \int\limits_a^b \frac{1}{y^2}dydx = \int\limits_c^d (\frac{1}{a} - \frac{1}{b})dx = (d - c)(\frac{1}{a} - \frac{1}{b})$.

Remark:
Note that not all the sides of A are geodesic in hyperbolic metric!



Christmas problems:

1. A Christmas tree on $\mathbf{H}^2$ is the union of the domains Δ_k, $k \in \mathbf{Z}$, where Δ_k is a Euclidean right-angled triangle with vertices
   $2^k i, 2^k(2i-1), 2^k(2i+1)$.
   (Note, it is not a hyperbolic triangle, its sides are not hyperbolic geodesics!)

   (a) Prove that $Vol(\Delta_k)$ does not depend on k.

   (b) Find $Vol(\Delta_k)$.

2. (*) Santa Claus (the hyperbolic one) decorates the tree with hyperbolic balls. He wants to arrange the decoration in such a way that the following two conditions are satisfied:

   i. Each Δ_k contains at least one ball;

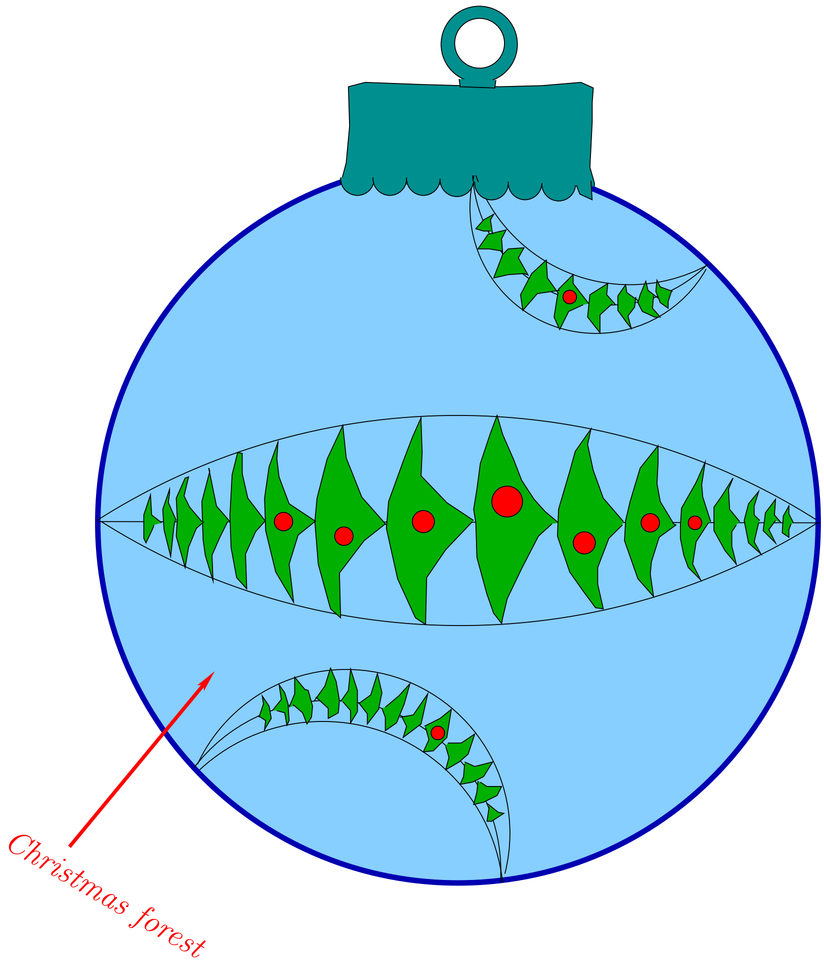
   ii. Volume of the union of all balls is finite.

   Can you suggest Santa an example of such a collection of balls?

   Remark: A hyperbolic ball of radius r centered at $z \in \mathbf{H}^2$ is the set of points on hyperbolic distance at most r from the point z.

   Hints:

   1. Show that balls centered in the center of the Poincaré disk model $\mathbf{B}^2$ are Euclidean balls.
   2. Use the isometry $-i\frac{z+1}{z-1} : \mathbf{B}^2 \to \mathbf{H}^2$ to see that balls centered at $i \in \mathbf{H}^2$ are Euclidean balls. Note that the hyperbolic center of a ball does not coincide with the Euclidean one.
   3. Use the isometries $z \to z + a$ and $z \to \alpha z$ ($a, \alpha \in \mathbf{R}$, $\alpha > 0$) to see that all balls in $\mathbf{H}^2$ are Euclidean balls.
   4. Let B be a ball centered at the center of the disk model $\mathbf{B}^2$. Find the volume of B as a function of its Euclidean radius r_E. What is its hyperbolic radius r? Find the volume of B as a function of its hyperbolic radius r.
   5. Take a family of balls $B_k \in \mathbf{B}^2$ of volume at most $\frac{1}{2^k}$ and use isometries to map these balls into $\Delta_{\pm k}$.
   6. What is the largest ball which fits into Δ_k?



Merry Christmas and Happy New Year!