Riemannian Geometry, Michaelmas 2013.

Homework 1

Starred problems due on Friday, October 25th

1. (*) Let M be a differentiable manifold of dimension m and N be a differentiable manifold of dimension n. Show that the cartesian product

$$M \times N := \{(x, y) \mid x \in M, y \in N\}$$

is a differentiable manifold of dimension $m + n$.

2. Let M^n be an n-dimensional smooth manifold. Show that there exists an atlas

$$\{(U_i \subseteq M^n, V_i \subseteq \mathbf{R}^n, \phi_i : U_i \to V_i), i \in I\}$$

inducing the same topology on M^n, such that V_i is the open unit ball in $\mathbf{R}^n$ for all $i \in I$.

3. Consider the *Lemniscate of Gerono* Γ, which is given as a subset of $\mathbf{R}^2$ by

$$\Gamma = \{(x, y) \in \mathbf{R}^2 \mid x^4 - x^2 + y^2 = 0\}.$$

You may google for a picture of this.

We give Γ a topology induced by its inclusion in $\mathbf{R}^2$ (setting the open subsets of Γ to be exactly those sets $\Gamma \cap U$ where U is an open subset of $\mathbf{R}^2$). Show that Γ with this topology does not admit the structure of a smooth 1-manifold.

4. For $a \in \mathbf{R}$ define the subset Γ_a of $\mathbf{R}^3$ by

$$(x, y, z) \in \Gamma_a \iff xyz = a,$$

(and give Γ_a a topology induced by inclusion in $\mathbf{R}^3$). For which values of a does Γ_a have the structure of a smooth 2-manifold?

5. This exercise shows that the matrix group $SL(n, \mathbf{R}) = \{A \in M(n, \mathbf{R}) \mid detA = 1\}$ is a differentiable manifold.

   (a) Let $f : \mathbf{R}^k \to \mathbf{R}$ be a homogeneous polynomial of degree $m \geq 1$. Prove *Euler's relation*

   $$\langle \operatorname{grad} f(x), x\rangle = mf(x),$$

   where

   $$\operatorname{grad} f(x) = \left(\frac{\partial f}{\partial x_1}(x), \frac{\partial f}{\partial x_2}(x), \ldots, \frac{\partial f}{\partial x_k}(x)\right).$$

   Hint: Differentiate $\lambda \mapsto f(\lambda x_1, \lambda x_2, \ldots, \lambda x_k)$ with respect to λ and use homogeneity.

   (b) Let $f : \mathbf{R}^k \to \mathbf{R}$ be a homogeneous polynomial of degree $m \geq 1$. Show that every value $y \neq 0$ is a regular value of f.

   (c) Use the fact that $\det A$ is a homogeneous polynomial in the entries of A in order to show that $SL(n, \mathbf{R})$ is a differentiable manifold.