Riemannian Geometry, Michaelmas 2013.

Homework 2

Starred problems due on Friday, October 25th

1. Show that a directional derivative is a derivation (i.e. check the Leibnic rule).

2. Let M be a differentiable manifold, $U \subset M$ open and $\varphi = (x_1, \dots, x_n) : U \to V_1 \subset \mathbf{R}^n$, $\psi = (y_1, \dots, y_n) : U \to V_2 \subset \mathbf{R}^n$ be two coordinate charts. Show for $p \in U$:
$$\frac{\partial}{\partial x_i}\Big|_p = \sum_{j=1}^n \frac{\partial(y_j \circ \varphi^{-1})}{\partial x_i}(\varphi(p)) \cdot \frac{\partial}{\partial y_j}\Big|_p,$$
where $y_j \circ \varphi^{-1} : V_1 \to \mathbf{R}$ and $\frac{\partial(y_j\circ\varphi^{-1})}{\partial x_i}$ is the classical partial derivative in the coordinate direction x_i of $\mathbf{R}^n$.

   Hint: Blow $f \circ \varphi^{-1}$ up to the expression $f \circ \psi^{-1} \circ \psi \circ \varphi^{-1}$, and apply the chain rule.

3. (*) This exercise is useful to become familiar with the notions introduced in the lectures. Let $S^2 = \{x \in \mathbf{R}^3 \mid \|x\| = 1\}$ be the standard two-dimensional sphere and $\mathbf{R}P^2$ be the real projective plane and $\pi : S^2 \to \mathbf{R}P^2$ be the canonical projection $p \mapsto p/\sim$. Let
$$c : (-\varepsilon, \varepsilon) \to S^2, \quad c(t) = (\cos t \cos(2t), \cos t \sin(2t), \sin t)$$
and
$$f : \mathbf{R}P^2 \to \mathbf{R}, \quad f(\mathbf{R}(z_1, z_2, z_3)^\top) = \frac{(z_1 + z_2 + z_3)^2}{z_1^2 + z_2^2 + z_3^2}.$$

   (a) Let $\gamma = \pi \circ c$. Calculate $\gamma'(0)(f)$.

   (b) Let (φ, U) be the following coordinate chart of $\mathbf{R}P^2$:
   $U = \{\mathbf{R}(z_1, z_2, z_3)^\top \mid z_1 \neq 0\} \subset \mathbf{R}P^2$ and
$$\varphi : U \to \mathbf{R}^2, \quad \varphi(\mathbf{R}(z_1, z_2, z_3)^\top) = \left(\frac{z_2}{z_1}, \frac{z_3}{z_1}\right).$$
   Let $\varphi = (x_1, x_2)$. Express $\gamma'(t)$ in the form
$$\alpha_1(t)\frac{\partial}{\partial x_1}\Big|_{\gamma(t)} + \alpha_2(t)\frac{\partial}{\partial x_2}\Big|_{\gamma(t)}.$$

4. The 3-sphere S^3 sits inside 2-dimensional complex space
$$S^3 = \{(w, z) \in \mathbf{C}^2 \ : \ |w|^2 + |z|^2 = 1.\}$$

   (a) Writing $w = a + ib$ and $z = c + id$ we can identify the tangent space to $\mathbf{C}^2 = \mathbf{R}^4$ at the point $(1, 0) \in \mathbf{C}^2$ with
$$< \partial/\partial a, \partial/\partial b, \partial/\partial c, \partial/\partial d > .$$
   In terms of this basis, what is the subspace tangent to S^3 at $(1, 0)$?

   (b) The map $\pi : S^3 \to \mathbf{C}$ given by $\pi(w, z) = z/w$ is defined away from $w = 0$. Identify the kernel of
$$D\pi : T_{(1,0)}S^3 \to T_0\mathbf{C}.$$

   By the way, if we identify $S^2 = \mathbf{C} \cup \{\infty\}$, then the map $\pi : S^3 \to S^2$ is defined everywhere and is smooth everywhere and is known as the *Hopf fibration* after the German mathematician Heinz Hopf.