Riemannian Geometry, Michaelmas 2013.

Homework 3-4

Starred problems due on Friday, November 8th

1. A Lie group $SO(n, \mathbf{R})$ is defined as $SO(n, \mathbf{R}) = \{A \subset M(n, \mathbf{R}) \mid A^\top A = I, detA = 1\}$.

   (a) Show that $SO(n, \mathbf{R})$ is a smooth manifold of dimension $\frac{n(n-1)}{2}$.
   Hint: use the Implicit Function Theorem.

   (b) Show that $SO(n, \mathbf{R})$ is a Lie group (with matrix multiplication as a group operation).

   (c) (*) Prove that the tangent space of the Lie group $SO(n) \subset M(n, \mathbf{R}) \cong \mathbf{R}^{n^2}$ at the identity $I \in SO(n)$ is given by
   $$T_I SO(n) = \{A \in M(n, \mathbf{R}) \mid A^\top = -A\},$$
   i.e., the space of all skew-symmetric $n \times n$-matrices.
   Hint: You may use that we have, componentwise, $(AB)'(t) = A'(t)B(t) + A(t)B'(t)$, for the product of any two matrix-valued curves.

2. Let M and N be smooth manifolds. Using local coordinates, explain why $T_{(p,q)}(M \times N) = T_pM \oplus T_qN$ for $p \in M$ and $q \in N$.

3. Let X and Y be two vector fields on $\mathbf{R}^3$ defined by
   $$\begin{aligned} X(x, y, z) &= z\frac{\partial}{\partial x} - 2z\frac{\partial}{\partial y} + (2y - x)\frac{\partial}{\partial z}, \\ Y(x, y, z) &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}. \end{aligned}$$
   And let S^2 sit inside $\mathbf{R}^3$ as the sphere of radius 1 centred at the origin.

   (a) Compute the Lie bracket $[X, Y]$.

   (b) Verify that the restrictions of the vector fields X and Y to S^2 are vector fields on S^2 (in other words, are everywhere tangent to S^2).

   (c) Check that the restriction of $[X, Y]$ to S^2 is also a vector field on S^2.

4. (*) Let M be a smooth manifold and let $X, Y, Z \in \mathfrak{X}(M)$ be vector fields on M, and let $a \in \mathbf{R}$. Prove the following identities concerning the Lie Bracket:

   (a) Linearity $[X + aY, Z] = [X, Z] + a[Y, Z]$.

   (b) Anti-symmetry $[Y, X] = -[X, Y]$,

   (c) The Jacobi Identity $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$.

   The Hairy Ball Theorem. Let S^n be an n-dimensional sphere $\sum_{i=1}^{n+1} x_i^2 = 1$ in $\mathbf{R}^{n+1}$. Let m be a positive integer. Then there does not exist a continuous non-vanishing vector field $X \in C^0(TS^{2m})$ on the even dimensional sphere S^{2m}

   This theorem tells us for example that it can not be windy everywhere at once on Earth's surface — at any given moment, the horizontal wind speed somewhere must be zero.

   The next exercise shows that The Hairy Ball Theorem does not hold for odd dimensions.

5. (a) Find vector fields $X, Y, Z \in \mathfrak{X}(S^3)$ such that $\{X(p), Y(p), Z(p)\}$ is a basis for T_pS^3 for all p S^3.

   (b) Can you find a non-vanishing vector field on TS^{2m+1} for arbitrary m?