Riemannian Geometry, Michaelmas 2013.

Homework 5

Starred problems due on Friday, November 22th

1. Let $M \subset \mathbf{R}^n$ be a smooth manifold given by the equation $f(x_1, \dots, x_n) = a$. Let $p \in M$ and $v \in T_pM$. Show that the vector $v = (v_1, \dots, v_n)$ satisfies the equation

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} v_i = 0.$$

Remark. Compare with Question 3 in Problems Class:
$M = \{(x_1, x_2, x_3) \in \mathbf{R}^3 \mid x_1^2 + x_2^2 = x_3\}$, we got vectors $(1, 0, 2x_1)$ and $(0, 1, 2x_2)$ as a basis of T_pM, and they do satisfy the equation $2x_1v_1 + 2x_2v_2 - v_3 = 0$!

2. (*) In class we established that the Poincare unit ball model of hyperbolic 2-space was isometric to the the upper half-plane model.

Now let $\mathbf{W}^2 = \{x \in \mathbf{R}^3 \mid q(x,x) = -1,\ x_3 > 0\}$ with $q(x,y) = x_1y_1 + x_2y_2 - x_3y_3$ be the hyperboloid model of the hyperbolic plane (see your notes for how we define the Riemannian metric on this space – it is not just the restriction of the standard Euclidean metric on $\mathbf{R}^3$). Also let the Poincare unit ball model B^2 of hyperbolic 2-space sit inside $\mathbf{R}^3$ as $\mathbf{B}^2 = \{x \in \mathbf{R}^3 \mid x_3 = 0,\ x_1^2 + x_2^2 < 1\}$.

We define a map $f : \mathbf{W}^2 \to \mathbf{B}^2$ by requiring that $p \in \mathbf{W}^2$ and $f(p) \in \mathbf{B}^2$ are collinear with the point $(0, 0, -1)$, for each $p \in \mathbf{W}^2$ (i.e. f is a projection from this point to the plane $\{z = 0\}$).

(a) Calculate explicitly the maps $f(X, Y, Z)$ for $(X, Y, Z) \in \mathbf{W}^2$ and $f^{-1}(x, y, 0)$ for $(x, y, 0) \in B^2$.
Hint: you will obtain

$$x = \frac{X}{Z+1}, \quad y = \frac{Y}{Z+1}.$$

and

$$f^{-1}(x,y) = \left(\frac{2x}{1 - x^2 - y^2}, \frac{2y}{1 - x^2 - y^2}, \frac{1 + x^2 + y^2}{1 - x^2 - y^2} \right).$$

(b) An almost global coordinate chart $\varphi : U \to V$ of $\mathbf{W}^2$ is given by

$$\varphi^{-1}(x_1, x_2) = (\cos(x_1)\sinh(x_2), \sin(x_1)\sinh(x_2), \cosh(x_2)),$$

where $0 < x_1 < 2\pi$ and $0 < x_2 < \infty$. Let $\psi = \varphi \circ f^{-1}$ be a coordinate chart for B^2 with coordinate functions y_1, y_2.
Calculate ψ^{-1} explicitly.

(c) Explain why

$$Df_p\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i},$$

for $i = 1, 2$ where $\frac{\partial}{\partial x_i} \in T_p\mathbf{W}^2$ and $\frac{\partial}{\partial y_i} \in T_{f(p)}\mathbf{B}^2$ for all $p \in U$.

(d) Show that

$$< \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} >_p = < \frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j} >_{f(p)}$$

for all $p \in U$, and $i, j \in \{1, 2\}$. Using the previous part of the question, this demonstrates that f is an isometry.

Additional remark. To be precise, we need to choose two coordinate charts of the above type with $V_1 = (0, 2\pi) \times (0, \infty)$ and $V_2 = (-\pi, \pi) \times (0, \infty)$ and an extra consideration for the linear map $Df(0,0,1) : T_{(0,0,1)}\mathbf{W}^2 \to T_0\mathbf{B}^2$ to cover all hyperbolic plane and to fully prove that f is an isometry.

3. Let $\mathbf{H}^2$ be the upper half-plane model of hyperbolic 2-space. We write $SL(2, \mathbf{R})$ for the 2×2 matrices with all real entries and determinant 1. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{R})$ and define the map

$$f_A : \mathbf{H}^2 \to \mathbf{H}^2, \; f_A(z) = \frac{az + b}{cz + d}.$$

It turns out that $f_A \circ f_B = f_{AB}$.

Show that the maps f_A are isometries of $\mathbf{H}^2$.

Hint: show first that

$$\mathrm{Im}(f_A(z)) = \frac{\mathrm{Im}(z)}{|cz + d|^2}.$$

4. Length calculations in the upper half plane model $\mathbf{H}^2$ of the hyperbolic plane.
   (a) Let $0 < a < b$ and $c : [a, b] \to \mathbf{H}^2$, $c(t) = ti$. Calculate the arc-length reparametrization $\gamma : [0, \ln(b/a)] \to \mathbb{H}^2$.
   (b) Let $c : [0, \pi] \to \mathbf{H}^2$, given by

   $$c(t) = \frac{ai \cos t + \sin t}{-ai \sin t + \cos t},$$

   for some $a > 1$. Calculate $L(c)$.

5. We work in the upper-half plane model of hyperbolic 2-space $\mathbf{H}^2$. We will show that for $z_1, z_2 \in \mathbf{H}^2$ the distance function is given by the formula

$$\sinh(\frac{1}{2}d(z_1, z_2)) = \frac{|z_1 - z_2|}{2\sqrt{\mathrm{Im}(z_1)\mathrm{Im}(z_2)}}.$$

   (a) Let $z_1 = iy_1$ and $z_2 = iy_2$ for $y_1, y_2 \in \mathbf{R}$, and verify that the formula holds in this case. (We derived the distance between two such points in class, you may use this result).
   (b) Let $A \in SL(2, \mathbf{R})$ and let $f_A(z)$ be the isometry of $\mathbf{H}^2$ considered Problem 2.
   Show that both sides of the formula are invariant under f_A (you may use the hint about $\mathrm{Im}(f_A(z))$ given Problem 2).
   (c) Finally, given two points $z_1, z_2 \in \mathbf{H}^2$, find an $A \in SL(2, \mathbf{R})$ such that both $f_A(z_1)$ and $f_A(z_2)$ lie on the imaginary axis.
   (d) Given what you know about Möbius transformations of $\mathbf{C}$, explain how you would draw the shortest path connecting two points $z_1, z_2 \in \mathbf{H}^2$.

Remark. If you took Geometry III last year, you may recall another formula

$$\cosh(d(z_1, z_2)) = 1 + \frac{|z_1 - z_2|^2}{2\mathrm{Im}(z_1)\mathrm{Im}(z_2)}.$$

How do these two formulas match each other?