Riemannian Geometry, Michaelmas 2013.

Homework 6

Starred problems due on Friday, November 22th

1. Show the properties of the covariant derivative in $\mathbf{R}^n$:

   (a) $\nabla_v(X+Y) = \nabla_v(X) + \nabla_v(Y)$

   (b) $\nabla_v(fX) = v(f)X(p) + f(p)\nabla_v X$

   (c) $\nabla_{\alpha v+\beta w}X = \alpha\nabla_v X + \beta\nabla_w X$

   (d) $v(\langle X, Y\rangle) = \langle\nabla_v X, Y\rangle + \langle X, \nabla_v Y\rangle$

   (e) $\nabla_X Y - \nabla_y X = [X, Y]$

   (here $X, V, v, w \in T_p\mathbf{R}^n$, $f \in C^\infty(\mathbf{R}^n)$ and $\alpha, \beta \in \mathbf{R}$).

2. (*) Let $\mathbf{H}^n$ be the upper-half plane model of hyperbolic n-space,

$$\mathbf{H}^n = \{x \in \mathbf{R}^n : x_n > 0\}, \quad g(v, w) = \frac{\langle v, w\rangle}{x_n^2},$$

   where we write g for the metric on $\mathbf{H}^n$ and we we identify each tangent space canonically with $\mathbf{R}^n$.

   Calculate all Christoffel symbols Γ_{ij}^k for the global coordinate chart just given by the identity map $\phi : \mathbf{H}^n \to \mathbf{R}^n$, $\phi(x) = x$.