Riemannian Geometry, Michaelmas 2013.

Homework 7–8

Starred problems due on Tuesday, December 3th

1. (*) Let $S^2 = \{(x,y,z) \in \mathbf{R}^3 \ : \ x^2+y^2+z^2 = 1\}$ be the unit sphere inside 3-space, with the induced metric from the standard Euclidean metric on $\mathbf{R}^3$.

   Let c be the curve on S^2 given by

$$c(t) = \left(\frac{1}{\sqrt{2}}, \frac{\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}} \right),$$

   and let $v \in T_{c(0)}S^2$ be given by

$$v = (0,1,0) \in T_{c(0)}S^2 \subset T_{c(0)}\mathbf{R}^3.$$

   Find $X(t) = (a_1(t), a_2(t), a_3(t))$ where $X(t) \in T_{c(t)}S^2 \subset T_{c(t)}\mathbf{R}^3$ is the unique (parallel) vector field along c determined by the parallel condition

$$\frac{D}{dt}X(t) = 0,$$

   and by

$$X(0) = v \in T_{c(0)}S^2.$$

2. Let $\mathbf{H}^2 = \{z \in \mathbf{C} \ : \ \mathrm{Im}(z) > 0\}$ be the upper-half plane with its usual hyperbolic metric. Let c be the curve in $\mathbf{H}^2$ given by $c(t) = i + t$ for $t \in \mathbf{R}$. Identifying the tangent space to each point of $\mathbf{H}^2$ in the usual way with $\mathbf{C}$, find the parallel vector field $X(t) \in \mathbf{C} = T_{c(t)}\mathbf{H}^2$ along c, which is determined by its value at $t = 0$:

$$X(0) = 1 \in \mathbf{C} = T_i\mathbf{H}^2.$$

3. Given a curve $c : [a,b] \to \mathbf{R}^3$, $c(t) = (f(t), 0, g(t))$ without self-intersections and with $f(t) > 0$ for all $t \in [a,b]$, let $M \subset \mathbf{R}^3$ denote the surface of revolution obtained by rotating this curve around the vertical Z-axis. Let ∇ denote the Levi-Civita connection of M. An almost global coordinate chart is given by $\varphi : U \to V := (a,b) \times (0, 2\pi)$,

$$\varphi^{-1}(x_1, x_2) = (f(x_1)\cos x_2, f(x_1)\sin x_2, g(x_1)).$$

   (a) Calculate the Christoffel symbols of this coordinate chart and express

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}$$

   in terms of the basis $\frac{\partial}{\partial x_k}$.

   (b) Let $\gamma_1(t) = \varphi^{-1}(x_1 + t, x_2)$. $\frac{D}{dt}$ denotes covariant derivative along γ_1. Calculate

$$\frac{D}{dt}\gamma_1'.$$

   Show that this vector field along γ_1 vanishes if and only if the generating curve c of M is parametrized proportional to arc-length. Note that γ_1 is obtained by rotation of c by a fixed angle. Derive from these facts that meridians of a surface of revolution are geodesics if they are parametrized proportional to arc length.

(c) Let $\gamma_2(t) = \varphi^{-1}(x_1, x_2 + t)$. $\frac{D}{dt}$ denotes covariant derivative along γ_2. Calculate

$$\frac{D}{dt}\gamma_2'.$$

Show that this vector field along γ_2 vanishes if and only if $f'(x_1) = 0$. Explain that this implies that parallels of a surface of revolution are geodesics if they have locally maximal or minimal radius.

4. Let M be a differentiable manifold, $\mathfrak{X}(M)$ be the vector space of smooth vector fields on M, and ∇ be a general affine connection (we do not require a Riemannian metric on M and the "Riemannian property", and also not the "torsionless property" of the Levi-Civita connection). We say, a map

$$A : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to C^\infty(M) \text{ or } \mathfrak{X}(M)$$

is a tensor, if it is linear in each argument, i.e.,

$$A(X_1, \cdots, fX_i + gY_i, \cdots, X_r) = fA(X_1, \cdots, X_i, \cdots, X_r) + gA(X_1, \cdots, Y_i, \cdots, X_r),$$

for all $X, Y \in \mathfrak{X}(M)$ and $f, g \in C^\infty(M)$.

(a) Show that

$$\begin{aligned} T : \mathfrak{X}(M) \times \mathfrak{X}(M) &\to \mathfrak{X}(M), \\ T(X,Y) &= [X,Y] - (\nabla_X Y - \nabla_Y X) \end{aligned}$$

is a tensor (called the "torsion" of the manifold M).

(b) Let

$$A : \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{r \text{ factors}} \to C^\infty(M)$$

be a tensor. The covariant derivative of A is a map

$$\nabla A : \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{r+1 \text{ factors}} \to C^\infty(M),$$

defined by

$$\nabla A(X_1, \ldots, X_r, Y) = Y(A(X_1, \ldots, X_r)) - \sum_{j=1}^{r} A(X_1, \ldots, \nabla_Y X_j, \ldots, X_r).$$

Show that ∇A is a tensor.

(c) Let (M, g) be a Riemannian manifold and $G : \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$ be the Riemannian tensor, i.e., $G(X,Y) = \langle X, Y \rangle$. Calculate ∇G. What does it mean that $\nabla G \equiv 0$?

5. Let (M, g) be a Riemannian manifold and $c : [a,b] \to M$ be a differentiable curve. Let $\frac{D}{dt}$ denote the corresponding covariant derivative along the curve c. Let X, Y be any two parallel vector fields X, Y along c. Show that

$$\frac{d}{dt}\langle X, Y \rangle \equiv 0,$$

i.e., the parallel transport $P_c : T_{c(a)}M \to T_{c(b)}M$ is a linear isometry.

(a) (*) Prove this statement in the particular case that the vector fields X, Y along c have global extensions $\tilde{X}, \tilde{Y} : M \to TM$.
Hint: use the Riemannian property of Levi-Civita connection.

(b) Do the same computation for a general case writing $X(t)$, $Y(t)$ in local coordinates.