Riemannian Geometry, Michaelmas 2013.

Homework 9–10

No due date for starred problems, unfortunatly...

1. **Rescaling Lemma.**
Let $c : [0, a] \to M$ be a geodesic, and $k > 0$. Define a curve γ by

$$\gamma : [0, a/k] \to M, \qquad \gamma(t) = c(kt)$$

Show that γ is geodesic with $\gamma'(t) = kc'(kt)$.

2. (*) **First Variation Formula of energy.**
Let $F : (-\varepsilon, \varepsilon) \times [a, b] \to M$ be a variation of a differentiable curve $c : [a, b] \to M$ with $c'(t) \neq 0$ for all $t \in [a, b]$ and X be its variational vector field. Let $E : (-\varepsilon, \varepsilon) \to \mathbf{R}_+$ denote the associated energy, i.e.,

$$E(s) = \frac{1}{2} \int_a^b \| \frac{\partial F}{\partial t}(s, t) \|^2 dt.$$

(a) Show that

$$E'(0) = \langle X(b), c'(b) \rangle - \langle X(a), c'(a) \rangle - \int_a^b \langle X(t), \frac{D}{dt} c'(t) \rangle dt.$$

Simplify the formula for the cases when

(b) c is a geodesic,

(c) F is a proper variation,

(d) c is a geodesic and F is a proper variation.

Let $c : [a, b] \to M$ be a curve connecting p and q (not necessarily parametrized proportional to arc length). Show that

(e) $E'(0) = 0$ for every proper variation implies that c is a geodesic.

(f) Assume that c minimizes the energy amongst all curves $\gamma : [a, b] \to M$ which connect p and q. Then c is a geodesic.

3. Let (M, g) be a Riemannian manifold and $p \in M$. Let $\varepsilon > 0$ be small enough such that

$$\exp_p : B_\varepsilon(0_p) \to B_\varepsilon(p) \subset M$$

is a diffeomorphism. Let $\gamma : [0,1] \to B_\varepsilon(p) \setminus \{p\}$ be any curve.

Show that there exist a curve $v : [0,1] \to T_pM$, $\| v(s) \| = 1$ for all $s \in [0,1]$, and a non-negative function $r : [0,1] \to \mathbf{R}_{\geq 0}$, such that

$$\gamma(s) = \exp_p(r(s)v(s)).$$

4. (*) (**Lemma 5.14.**) Use the exponential map to show that any vector field along a smooth curve $c(t) : [a,b] \to M$ is a variation vector field of some variation $F(s,t)$. Show that if $X(a) = X(b) = 0$ then the variation $F(s,t)$ may be chosen among proper veriations.

5. **Geodesic normal coordinates.**
Let (M, g) be a Riemannian manifold and $p \in M$. Let $\varepsilon > 0$ such that

$$\exp_p : B_\varepsilon(0_p) \to B_\varepsilon(p) \subset M$$

is a diffeomorphism. Let $v_1, \ldots, v_n$ be an orthonormal basis of T_pM. Consider a local coordinate chart of M given by $\varphi = (x_1, \ldots, x_n) : B_\varepsilon(p) \to V := \{w \in \mathbb{R}^n \mid \|w\| < \varepsilon\}$ via

$$\varphi^{-1}(x_1, \ldots, x_n) = \exp_p(\sum_{i=1}^n x_i v_i).$$

The coordinate functions $x_1, \ldots, x_n$ of φ are called *geodesic normal coordinates.*

(a) Let g_{ij} be the first fundamental form in terms of the above coordinate system φ. Show that at $p \in M$:

$$g_{ij}(p) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

(b) Let $w = (w_1, \ldots, w_n) \in \mathbb{R}^n$ be arbitrary vector, and $c(t) = \varphi^{-1}(tw)$. Explain why $c(t)$ is a geodesic and deduce from this fact that

$$\sum_{i,j} w_i w_j \Gamma_{ij}^k(c(t)) = 0,$$

for all $1 \leq k \leq n$.

(c) Derive from (b) that all Christoffel symbols Γ_{ij}^k of the chart φ vanish at the point $p \in M$.