Riemannian Geometry, Michaelmas 2013.

Term 1: outline

1 Smooth manifolds

"Smooth" means "infinitely differentiable", C^∞.

Definition 1.1. Let M be a set. An n-dimensional smooth atlas on M is a collection of triples $(U_\alpha, V_\alpha, \varphi_\alpha)$, where $\alpha \in I$ for some indexing set I, s.t.

0. $U_\alpha \subseteq M$; $V_\alpha \subseteq \mathbf{R}^n$ is open;
1. $\bigcup_{\alpha\in I} U_\alpha = M$;
2. Each $\varphi_\alpha : U_\alpha \to V_\alpha$ is a bijection;
3. The composition $\varphi_\beta \circ \varphi_\alpha^{-1} \mid_{\varphi_\alpha(U_\alpha \cap_\beta)}$: $\varphi_\alpha(U_\alpha \cap_\beta) \to \varphi_\beta(U_\alpha \cap_\beta)$ is a smooth map for all ordered pairs (α, β), where $\alpha, \beta \in I$.

The number n is called the dimension of M, the maps φ_α are called coordinate charts, the compositions $\varphi_\beta \circ \varphi_\alpha^{-1}$ are called transition maps or change of coordinates.

Definition 1.2. M is called a smooth n-dimensional manifold if

1. M has an n-dimensional smooth atlas;
2. M is Hausdorff (see Def. 1.4 below);
3. M is second-countable (technical condition, we will ignore).

Definition 1.3. Let M have a smooth atlas. We call a set $A \subseteq M$ open iff for each $\alpha \in I$ the set $\varphi_\alpha(A \cap U_\alpha)$ is open in $\mathbf{R}^n$. This defines a topology on M.

Definition 1.4. M is called Hausdorff if for each $x, y \in M$, $x \neq y$ there exist open sets A_x and A_y such that $x \in A_x$, $y \in A_y$ and $A_x \cap A_y = \emptyset$.

Example 1.6. Examples of smooth manifolds: sphere, torus, Klein bottle, 3-torus, real projective space.

Definition 1.7. Let $f : M^m \to N^n$ be a map of smooth manifolds with atlases $(U_i, \varphi_i(U_i), \varphi_i)_{i\in I}$ and $(U_j, \psi_j(U_j), \psi_j)_{j\in J}$. The map f is smooth if it induces smooth maps between the open sets in $\mathbf{R}^m$ and $\mathbf{R}^n$, i.e. if $\psi_j \circ f \circ \varphi_i^{-1} \mid_{\varphi_i(f^{-1}(V_j \cap f(U_i)))}$ is smooth for all $i \in I$, $j \in J$.
If f is a bijection and both f and f^{-1} are smooth then f is called a diffeomorphism.

Definition 1.8. Let $U \subseteq \mathbf{R}^n$ be open, $m < n$ and $f : U \to \mathbf{R}^m$ be a smooth map. Let $Df \mid_x = (\frac{\partial f_i}{\partial x_j})$ be the matrix of partials at $x \in U$ (differential). Then

- $x \in \mathbf{R}^n$ is a regular point of f if $rk(Df \mid_x) = m$;
- $y \in \mathbf{R}^m$ is a regular value of f if $f^{-1}(\{y\})$ consists of regular points only.

Theorem 1.9. (Implicit Function Theorem).
If $y \in f(U)$ is a regular value of f then $f^{-1}(y)$ is an $(n-m)$-dimensional smooth manifold.

Examples 1.10-1.11. An ellipsoid is a smooth manifold. Matrix groups are smooth manifolds.

Definition 1.12. A Lie group is a smooth manifold G together with a group operation
$G \times G \to G$ s.t. the maps $(g_1, g_2) \to g_1 \cdot g_2$ and $g \to g^{-1}$ are smooth.
In particular, all matrix groups are Lie groups.

2 Tangent space

Definition 2.1. Let M be a smooth manifold, $p \in M$. Then $C^\infty(M,p)$ is a set of all smooth functions on M defined in a neighbourhood of p.

Definition 2.2. A derivation on $C^\infty(M,p)$ is a linear map $\delta : C^\infty(M,p) \to \mathbf{R}$, s.t.
for all $f, g \in C^\infty(M,p)$ holds $\quad \delta(f \cdot g) = f(p)\delta(g) + \delta(f)g(p) \quad$ (the Leibniz rule).
Denote by $\mathcal{D}^\infty(M,p)$ the set of all derivations. Check that it is a real vector space.

Definition 2.3. The set $\mathcal{D}^\infty(M,p)$ is called the tangent space of M at p, denoted T_pM.

Definition 2.4. Let $\gamma : (a,b) \to M$ be a smooth curve in M, $\gamma(t_0) = p$ and $f \in C^\infty(M,p)$. Define the directional derivative of f at p along γ by $\gamma'(t_0)(f) \in \mathbf{R}$:

$$\gamma'(t_0)(f) = \lim_{s\to 0} \frac{f(\gamma(t_0+s)) - f(\gamma(t_0))}{s} = (f \circ \gamma)'(t_0) = \frac{d}{dt}\Big|_{t=t_0}(f \circ \gamma)$$

Check that the directional derivatives satisfy the properties of derivations.

Remark. Two curves γ_1 and γ_2 through p define the same directional derivative iff they have the same direction and the same speed at p.

Notation. Let M^n be a manifold, $\varphi : U \to V \subseteq \mathbf{R}^n$ a chart at $p \in U \subset M$. For $i = 1, \dots, n$ define the curves $\gamma_i(t) = \varphi^{-1}(\varphi(p) + e_i t)$ for small $t > 0$ (here e_i is a basis of $\mathbf{R}^n$).

Def. 2.5. $\frac{\partial}{\partial x_i}\big|_p := \gamma_i'(0)$, i.e.

$$\frac{\partial}{\partial x_i}\Big|_p(f) = (f \circ \gamma_i)'(0) = \frac{d}{dt}(f \circ \varphi^{-1})(\varphi(p) + te_i)\Big|_{t=0} = \frac{\partial}{\partial x_i}(f \circ \varphi^{-1})(\varphi(p))$$

($\frac{\partial}{\partial x_i}$ on the right is just a classical partial derivative).

Proposition 2.6. $\langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \rangle = \{\text{Directional Derivatives}\} = \mathcal{D}^\infty(M,p)$.

Lemma 2.7. Let $\varphi : U \subseteq M \to \mathbf{R}^n$ be a chart, $\varphi(p) = 0$. Let $\tilde{\gamma}(t) = \sum_{i=1}^n (k_i t e_i) \in \mathbf{R}^n$ be a curve (straight ray), where $\langle e_1, \dots, e_n \rangle$ is a basis. Let $\gamma(t) = \varphi^{-1}(t) \in M$, $p \in \gamma(0)$. Then $\gamma'(0) = \sum_{i=1}^n k_i \frac{\partial}{\partial x_i}$.

Example 2.8. For the group $SL(n,\mathbf{R}) = \{A \in M_n \mid detA = 1\}$, the tangent space at I is the set of all trace-free matrices: $T_I(SL(n,\mathbf{R})) = \{X \in M_n(\mathbf{R}) \mid trX = 0\}$.

Proposition 2.9. (Change of basis for T_pM). Let M^n be a smooth manifold, $\varphi_\alpha : U_\alpha \to V_\alpha$ a chart, $(x_1^\alpha, \dots, x_n^\alpha)$ the coordinates in V_α. Then $\frac{\partial}{\partial x_j^\alpha}\ |_p = \sum_{i=1}^n \frac{\partial x_i^\beta}{\partial x_j^\alpha} \frac{\partial}{\partial x_i^\alpha}$.

Definition 2.10. Let M, N be smooth manifolds, let $f : M \to N$ be a smooth map. Define a linear map $Df(p) : T_pM \to T_{f(p)N}$ called the differential of f at p by $Df(p)\gamma'(0) = (f \circ \gamma)'(0)$ for a smooth curve $\gamma \in M$ with $\gamma(0) = p$.

Lemma 2.11. (a) $D(id)(p) : T_pM \xrightarrow{I} T_pM$;
(b) for $M \xrightarrow{g} N \xrightarrow{f} L$ holds $D(g \circ f)\big|_p = Dg\big|_{f(p)} \circ Df\big|_p$.

Tangent bundle and vector fields

Definition 2.13. Let M be a smooth manifold. A disjoint union $TM = \cup_{p\in M} T_pM$ of tangent spaces to each $p \in M$ is called a tangent bundle.
There is a map $\Pi : TM \to M$ (called projection), $\Pi(v) = p$ if $v \in T_pM$.

Proposition 2.14. The tangent bundle TM if M^n has a structure of $2n$-dimensional smooth manifold and $\Pi : TM \to M$ is a smooth map.

Definition 2.15. A vector field X is a "section" of the tangent bundle, that is a smooth map $X : M \to TM$ such that $\Pi \circ X = id_M$ is an identity map on M.

The set of all vector fields on M is denoted $\mathfrak{X}(M)$. This set has a structure of a vector space.

Remark 2.16. Taking a coordinate chart $(U, \varphi = (x_1, \dots, x_n)))$ we can write any vector field $X\mid_U$ as $X(p) = \sum_{i=1}^n f_i(p)\frac{\partial}{\partial x_i} \in T_pM$.

Examples 2.17-2.18: vector fields on the torus and 3-sphere.

Remark 2.19. Since for $X \in \mathfrak{X}(M)$ we have $X_p \in T_pM$ which is a directional derivative at $p \in M$, we can use the vector field to differentiate a function $f \in C^\infty(M)$, $f : M \to \mathbf{R}$ by $(Xf)(p) = X(p)f = \sum a_i(p)\frac{\partial f}{\partial x_i}\big|_p$, so that we get another smooth function $X_f \in C^\infty(M)$.

Proposition 2.20. Let $X, Y \in \mathfrak{X}(M)$. Then there exists a unique vector field $Z \in \mathfrak{X}(M)$ such that $Z(f) = X(Y(f)) - Y(X(f))$ for all $f \in C^\infty(M)$.

This vector field $Z(f) = X(Y(f)) - Y(X(f))$ is denoted $[X, Y]$ and called the Lie bracket.

Proposition 2.21. Properties of the Lie bracket:

a. $[X, Y] = -[Y, X]$;

b. $[aX + bY, Z] = a[X, Z] + b[Y, Z]$ for $a, b \in \mathbf{R}$;

c. $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ (Jacobi identity);

d. $[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$ for $f, g \in C^\infty(M)$.

Definition 2.22. A Lie algebra is a vector space $\mathfrak{g}$ with a binary operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the Lie bracket which satisfies properties a,b,c of Proposition 2.21.
In other words, $\mathfrak{X}(M)$ is a Lie algebra.

Theorem 2.23. (The Hairy Ball Theorem).
There is no non-vanishing continuous vector field on an even-dimensional sphere S^{2m}.

3 Riemannian metric

Definition 3.1. Let M be a smooth manifold. A Riemannian metric written $g_p(\cdot,\cdot)$ or $\langle\cdot,\cdot\rangle_p$ is a family of real inner products $g_p : T_pM \times T_pM \to \mathbf{R}$ depending smoothly on $p \in M$.
A smooth manifold M with a Reimannian metric g is called a Riemannian manifold (M, g).

Examples 3.2–3.3. Euclidean metric in $\mathbf{R}^n$, induced metric on $M \in \mathbf{R}^n$.

Definition 3.4. Let (M, g) be a Riemannian manifold. For $v \in T_pM$ define the length of v by $0 \le ||v||_g = \sqrt{g_p(v, v)}$.
Suppose $c : [a, b] \to M$ is a smooth curve on M. Define the length of c by $L(c) = \int_a^b ||c'(t)||dt$.
(this does not depend on parametrization, see Theorem 3.10).

Remark 3.5. Let $M \in \mathbf{R}^n$ be a smooth manifold given by $f(x_1, \dots, x_n) = a$. Let $p \in M$, $v \in T_pM$. Then v satisfies $\sum_{i=1}^n \frac{\partial f}{\partial x_i} v_i = 0$.

Example 3.6. three models of hyperbolic geometry:

model	notation	M	g
Hyperboloid	$\mathbf{W}^n$	$\{y \in \mathbf{R}^{n+1} \mid q(y, y) = -1, y_{n+1} > 0\}$ where $q(x, y) = \sum_{i=1}^n x_i y_i - x_{n+1}y_{n+1}$	$\langle v, w\rangle = q(v, w)$
Poincaré ball	$\mathbf{B}^n$	$\{x \in \mathbf{R}^n \mid ||x||^2 = \sum\limits_{i=1}^n x_i^2 < 1\}$	$g_x(v, w) = \frac{4}{(1-||x||^2)^2}\langle v, w\rangle$
Upper half-space	$\mathbf{H}^n$	$\{x \in \mathbf{R}^n \mid x_n > 1\}$	$g_x(v, w) = \frac{1}{x_n^2}\langle v, w\rangle$

Definition 3.7. Given two vector spaces V_1, V_2 with real inner products $(V_i, \langle\cdot,\cdot\rangle_i)$ we call a linear isomorphism $T : V_1 \to V_2$ a linear isometry if $\langle Tv, Tw\rangle_2 = \langle v, w\rangle_1$ for all $v, w \in V_1$.
This is equivalent to preservation of the lengths of all vectors since $\langle v, w\rangle = \frac{1}{2}(\langle v+w, v+w\rangle - \langle v, v\rangle - \langle w, w\rangle)$.

A diffeomorphism $f : (M, g) \to (N, h)$ of two Riemannian manifolds is an isometry if $DF(p) : T_pM \to T_{f(p)N}$ is a linear isometry for all $p \in M$.

Example 3.9. Isometry $f : \mathbf{H}^2 \to \mathbf{B}^2$ given by $f(z) = \frac{z-i}{z+i}$.

Theorem 3.10. (Reparametrization). Let $\varphi : [c, d] \to [a, b]$ be a strictly monotonic smooth function, $\gamma : [a, b] \to M$ a smooth curve. Then for $\tilde{c} = c \circ \varphi : [c, d] \to M$ (reparametrization of c) holds $L(c) = L(\tilde{c})$.

Definition 3.11. A differentiable curve $c : [a, b] \to M$ is called an arc-length parametrization if $||c'(t)|| = 1$. Every curve has an arc-length parametrization $\tilde{c}(t) : [0, L(c|_{[a,b]})] \to M$, $\tilde{c}(t) = c \circ \varphi(t)$, where $\varphi^{-1}(t) = L(c|_{[a,t]})$.

Example 3.12. Length of vertical segments in $\mathbf{H}$. Vertical half-lines are geodesics.

Definition 3.13. Define a distance $d : M \times M \to [0, \infty)$ by $d(p, q) = inf_\gamma\{L(\gamma)\}$, where γ is a smooth curve with end p and q.
A curve $c(t) : [a, b] \to M$ is geodesic if $d(c(x), c(y)) = L(c|_{[x,y]})$ for all $x, y \in [a, b]$ $(x < y)$.

Remark 3.14. d turns (M, g) into a metric space.

Definition 3.15. If (X, d) is a metric space then any subset $A \in X$ is also a metric space with the induced metric $d|_{A\times A} : A \times S \to [0, \infty)$.

Example 3.16 Punctured Riemannian sphere: $\mathbf{R}^n$ with $g_x(v, w) = \frac{4}{(1+||x||^2)^2}\langle v, w\rangle$.

4 Levi-Civita connection and parallel transport

4.1 Levi-Civita connection

Example 4.1 In $\mathbf{R}^n$, given a vector field $X = \sum a_i(p)\frac{\partial}{\partial x_i} \in \mathfrak{X}(\mathbf{R}^n)$ and a vector $v \in T_p\mathbf{R}^n$ define the covariant derivative of X in direction v by $\nabla_v(X) = \lim\limits_{t\to 0} \frac{X(p+tv)-X(p)}{t} = \sum v(a_i)\frac{\partial}{\partial x_i}\big|_p \in T_p\mathbf{R}^n$.

Properties 4.2. In $\mathbf{R}^n$, the covariant derivative $\nabla_v X$ satisfies properties (a)-(e) listed below in Definition 4.3 and Theorem 4.4.

Definition 4.3. Let M be a smooth manifold. A map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, $X, Y \nabla_X Y$ is called a covariant derivative or affine connection if for all $X, Y, Z \in \mathfrak{X}(M)$ and $f, g \in C^\infty(M)$ holds

(a) $\nabla_X(Y + Z) = \nabla_X(Y) + \nabla_X(Z)$

(b) $\nabla_X(fY) = X(f)Y(p) + f(p)\nabla_X Y$

(c) $\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z$

Theorem 4.4. (Levi-Civita, Fundamental Theorem of Riemannian Geometry). Let (M, g) be a Riemannian manifold. There exists a unique covariant derivative ∇ on M with the additional properties for all $X, Y, Z \in \mathfrak{X}(M)$:

(d) $v(\langle X, Y\rangle) = \langle\nabla_v X, Y\rangle + \langle X, \nabla_v Y\rangle$ (Riemannian property);

(e) $\nabla_X Y - \nabla_Y X = [X, Y]$ (Torsion-free)

This connection is called Levi-Civita connection of (M, g).

Example 4.5. Levi-Civita connection in $\mathbf{R}^n$ and in $M \subset \mathbf{R}^n$ with induced metric.

4.2 Christoffel symbols

Definition 4.6. Let ∇ be a Levi-Civita connection on (M,g) and $\varphi: U \to V$ a coordinate chart with coordinates $\varphi = (x_1,\dots,x_n)$. Then we have $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}(p) \in T_pM$. i.e. there exist uniquely determined $\Gamma_{ij}^k \in C^\infty(U)$ with $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}(p) = \sum_{k=1}^n \Gamma_{ij}^k \frac{\partial}{\partial x_k}(p)$. These functions are called Christoffel symbols of ∇ with respect to the chart φ.

They characterize ∇ since $\qquad \nabla_{\sum\limits_{i=1}^n a_i \frac{\partial}{\partial x_i}} \sum\limits_{j=1}^n b_j \frac{\partial}{\partial x_j} = \sum\limits_{i,j} a_i \frac{\partial b_j}{\partial x_i} \frac{\partial}{\partial x_j} + \sum\limits_{i,j,k} a_i b_j \Gamma_{i,j}^k \frac{\partial}{\partial x_k}$.

Proposition 4.7. $\Gamma_{ij}^s = \frac{1}{2} \sum\limits_k g^{ks}(g_{ik,j} + g_{jk,i} - g_{ij,k})$, where $g_{ab,c} = \frac{\partial}{\partial x_c} g_{ab}$ and $(g^{ij}) = (g_{ij})^{-1}$.

In particular, $\Gamma_{ij}^k = \Gamma_{ji}^k$.

Example 4.8. In $\mathbf{R}^n$, $\Gamma_{ij}^k \equiv 0$ for all i,j,k; Γ_{ij}^k in $S^2 \subset \mathbf{R}^3$ with induced metric.

4.3 Parallel transport

Definition 4.9. Let $c:(a,b) \to M$ be a differentiable curve. A map $X:(a,b) \to TM$ with $X(t) \in T_{c(t)}M$ is called a vector field along c. Denote the space of all these maps by $\mathfrak{X}_c(M)$.

Example 4.10. $c'(t) \in \mathfrak{X}_c(M)$.

Proposition 4.11. Let (M,g) be a Riemannian manifold, ∇ be a Levi-Civita connection, $c:(a,b) \to M$ be a differentiable curve. There exists a unique map $\frac{D}{dt}: \mathfrak{X}_c(M) \to \mathfrak{X}_c(M)$ satisfying

(a) $\frac{D}{dt}(X+Y) = \frac{D}{dt}X + \frac{D}{dt}Y$

(b) $\frac{D}{dt}(fX) = f'(t)X + f\frac{D}{dt}X$, for all differentiable $f:(a,b) \to \mathbf{R}$

(c) If $\widetilde{X} \in \mathfrak{X}(M)$ is a local extension of X
(i.e. there exists $t_0 \in (a,b)$ and $\varepsilon > 0$ such that $X(t) = \widetilde{X}\big|_{c(t)}$ for all $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$)
then $(\frac{D}{dt}X)(t_0) = \nabla_{c'(t_0)}\widetilde{X}$.

This map $\frac{D}{dt}: \mathfrak{X}_c(M) \to \mathfrak{X}_c(M)$ is called the covariant derivative along the curve c.

Example 4.12. For a surface $M \in \mathbf{R}^3$ the condition $\frac{D}{dt}(c'(t)) \equiv 0$ is equivalent to $c''(t) \perp T_{c(t)}M$, which is in its turn the condition for c to be geodesic known from the course of Differential Geometry.

Definition 4.13. Let $X \in \mathfrak{X}_c(M)$. If $\frac{D}{dt}X = 0$ then X is said to be parallel along c.

Example 4.14. In $\mathbf{R}^n$ it means that X does not depend on the point $p \in \mathbf{R}^n$.

Theorem 4.15. Let $c:[a,b] \to M$ be a smooth curve, $v \in T_{c(a)M}$. Then there exists a unique vector field $X \in \mathfrak{X}_c(M)$ with $X(a) = v \in T_{c(a)}M$.

Example 4.16. Parallel vector fields form a vector space of dimension n (for n-dimensional (M,g)).

Definition 4.17. Let $c:[a,b] \to M$ be a smooth curve, A linear map $P_c: T_{c(a)}M \to T_{c(b)}M$ called parallel transport defined by $P_c(v) = X(b)$ where $X \in \mathfrak{X}_c(M)$ with $X(a) = v$, $\frac{D}{dt}X = 0$.

Remark. (a) The parallel transport P_c depend on the curve c (not only on its endpoints).
(b) The parallel transport is a linear isometry $P_c: T_{c(a)}M \to T_{c(b)}M$, i.e. $g_{c(a)}(v,w) = g_{c(b)}(P_cv, P_cw)$.

5 Geodesics

5.1 Geodesics as solutions to ODEs

Definition 5.1. Given (M, g), the curve $c : [a, b] \to M$ is a geodesic if $\frac{D}{dt}c'(t) = 0$ for all $t \in [a, b]$.

Lemma 5.2. If c is a geodesic that c is parametrized proportional to the arc length.

Theorem 5.3. Given a Riemannian manifold (M, g), $p \in M$, $v \in T_pM$, there exists $\varepsilon > 0$ and a unique geodesic $c : (-\varepsilon, \varepsilon) \to M$ such that $c(0) = p$, $c'(0) = v$.

Example 5.4–5.5. Geodesics in Euclidean space, on a sphere, and in the upper half-plane model $\mathbf{H}^2$.

Remark. Differential equations for geodesics: $c_k''(t) = -\sum_{ij} c_i'(t)c_j'(t)\Gamma_{ij}^k(c(t))$, $k = 1, \dots, n$.

5.2 Geodesics as distance-minimizing curves. First variation formula of the length.

Definition 5.6. Let $c : [a, b] \to M$ be a smooth curve. A smooth map $F : (-\varepsilon, \varepsilon) \times [a, b] \to M$ is a differentiable variation of c if $F(0, t) = c(t)$.
Variation is proper if $F(s, a) = c(a)$ and $F(s, b) = c(b)$ for all $s \in (-\varepsilon, \varepsilon)$.
Variation may be considered as a family of the curves $F_s(t) = F(s, t)$.

Definition 5.7. A variation vector field X of the variation F is defined by $X(t) = \frac{\partial F}{\partial s}(0, t)$.

Definition 5.8. The length and energy of variation are

$l(s) := \int_a^b ||\frac{\partial F}{\partial t}(s, t)||dt, \quad l : (-\varepsilon, \varepsilon) \to [0, \infty);$ $\qquad E(s) := \int_a^b ||\frac{\partial F}{\partial t}(s, t)||^2 dt, \quad E : (-\varepsilon, \varepsilon) \to [0, \infty).$

Remark: $l(s)$ is the length of the curve $F_s(t)$.

Theorem 5.9. A smooth curve c is geodesic if and only if $l'(0) = 0$ for each proper variation and c is parametrized proportionally to the arc length.

Corollary 5.10. Let $c : [a, b] \to M$ be the shortest curve from $c(a)$ to $c(b)$, and c is parametrized proportionally to the arc length. Then c is geodesic.

Lemma 5.11. (Symmetry Lemma). Let $W \subset \mathbf{R}^2$ be an open set and $F : W \to M$, $(s, t) \mapsto F(s, t)$ be a differentiable map. Let $\frac{D}{dt}$ be the covariant derivative along $F_s(t)$ and $\frac{D}{ds}$ be the covariant derivative along $F_t(s)$. Then $\frac{D}{dt}\frac{\partial F}{\partial s} = \frac{D}{ds}\frac{\partial F}{\partial t}$.

Theorem 5.12. (First variation formula of the length). Let $F : (-\varepsilon, \varepsilon) \times [a, b] \to M$ be a variation of $c(t)$, $c'(t) \neq 0$. Let $X(t)$ be its variation vector field and $l : (-\varepsilon, \varepsilon) \to [0, \infty)$ its length. Then

$$l'(0) = \int_a^b \frac{1}{||c'(t)||}\frac{d}{dt}\langle X(t), c'(t)\rangle dt - \int_a^b \frac{1}{||c'(t)||}\langle X(t), \frac{D}{dt}c'(t)\rangle dt.$$

Corollary 5.13.

- If in addition $c(t)$ is parametrized proportionally to the arc length, $||c'(t)|| \equiv c$ then $l'(0) = \frac{1}{c}\langle X(b), c'(b)\rangle - \frac{1}{c}\langle X(a), c'(a)\rangle - \frac{1}{c}\int_a^b \langle X(t), \frac{D}{dt}c'(t)\rangle dt$;
- if $c(t)$ is geodesic, then $l'(0) = \frac{1}{c}\langle X(b), c'(b)\rangle - \frac{1}{c}\langle X(a), c'(a)\rangle$;
- if F is proper and c is parametrised proportionally to the arc length, then $l'(0) = -\frac{1}{c}\int_a^b \langle X(t), \frac{D}{dt}c'(t)\rangle dt$;
- if F is proper and c is geodesic, then $l'(0) = 0$.

Lemma 5.14. Any vector field X along $c(t)$ with $X(a) = X(b) = 0$ is a variation vector field for some proper variation F.

5.3 Exponential map and Gauss Lemma

Let $p \in M$, $v \in T_pM$. Denote by $c_v(t)$ the unique maximal (by inclusion) geodesic with $c_v(0) = p$, $c_v'(0) = v$.

Definition 5.15. If $c_v(1)$ exists, define $exp_p : T_pM \to M$ by $v \mapsto c_v(1)$, the exponential map at p.

Example 5.16. Exponential map on the sphere S^2: length of c_v from p to $c_v(1)$ equals to $||v||$.

Notation. $B_r(O_p) = \{v \in T_pM \mid ||v|| < r\} \subset T_pM$ is a ball of radius r centered at $p = O_p$.

Proposition 5.17. (without proof).
For any $p \in (M, g)$ there exists an $r > 0$ such that $exp_p : B_r(O_p) \to exp_p(B_r(O_p))$ is a diffeomorphism.

Example. On S^2 the set $exp_p(B_{\pi/2}(O_p))$ is a hemisphere, so that every geodesic starting from p is orthogonal to the boundary of this set.

Theorem 5.18. (Gauss Lemma). Let (M, g) be a Riemannian manifold, $p \in M$ and let $\varepsilon > 0$ be such that $exp_p : B_\varepsilon(O_p) \to exp_p(B_\varepsilon(O_p))$ is a diffeomorphism. Define a hypersurface $A_\delta := \{exp_p(w) \mid ||w|| = \delta\}$ for all $0 < \delta < \varepsilon$. Then every radial geodesic $c : t \to exp_p(tv)$, $t \geq 0$ is orthogonal to A_δ.

Remark 5.19. The curve $c_v(t) = exp_p(tv)$ is indeed a geodesic!

Lemma 5.19. Let $p \in M$ and let $\varepsilon > 0$ be such that $exp_p : B_\varepsilon(O_p) \to exp_p(B_\varepsilon(O_p))$ is a diffeomorphism. Take $\gamma : [0, 1] \to exp_p(B_\varepsilon(O_p))$. Then there exists a curve $v(s) : [0, 1] \to T_p(M)$, $||v(s)|| = 1$ and a non-negative function $r(s) : [0, 1] \to \mathbf{R}_+$ such that $\gamma(s) = exp_p(r(s) \cdot v(s))$.

Lemma 5.20. Let $r : [0, 1] \to \mathbf{R}$, $v : [0, 1] \to S_pM = w \in T_pM \mid ||w|| = 1\}$. Define $\gamma : [0, 1] \to exp_p(B_\varepsilon(O_p))$ by $\gamma(s) = exp_p(r(s)v(s))$. Then $l(\gamma) \geq |r(1) - r(0)|$ for the length $l(\gamma)$ of γ and the equality holds if and only if γ is a reparametrisation of a radial geodesic (i.e. if and only if $v(s) \equiv const = v(0)$, $r(s)$ is increasing or decreasing function).

Corollary 5.21. Given a point $p \in M$, there exists $\varepsilon > 0$ such that for any $q \in B_\varepsilon(O_p)$ there exists a curve $c(t)$ connecting p and q and satisfying $l(c) = d(p, q)$. (This curve is a radial geodesic).

Notation. Denote $B_\varepsilon(p) := exp_p(B_\varepsilon(O_p)) \subset M$, a geodesic ball and $S_\varepsilon(p) = \partial B_\varepsilon(p)$, a geodesic sphere.
Note, $B_\varepsilon(p) = \{q \in M \mid d(p, q) \leq \varepsilon\}$.

Proposition 5.22. (without proof). Let $p \in M$. Then there exists an open set U_p, $p \in U$ and an ε such that for any $q \in U$ the map $exp_q : B_\varepsilon(O_q) \to B_\varepsilon(q)$ are diffeomorphisms.

Remark 5.23. (Naturality of exponential map).
Let $\varphi : (M, g) \to (N, h)$ be an isometry. Then $D\varphi = exp^{-1}_{\varphi(p)} \circ \varphi \circ exp_p$.

5.4 Hopf-Rinow Theorem

Definition 5.24. A geodesic $c : [a, b] \to M$ is minimal if $l(c) = d(c(a), c(b))$.
A geodesic $c : \mathbf{R} \to M$ is minimal if its restriction $c|_{[a,b]}$ is minimal for each segment $[a, b] \in \mathbf{R}$.

Example: no minimal geodesics on S^2, all geodesics are minimal in $\mathbf{E}^2$ and $\mathbf{H}^2$.

Definition 5.25. A Riemannian manifold (M, g) is geodesically complete if every geodesic $c : [a, b] \to M$ can be extended to a geodesic $\tilde{c} : \mathbf{R} \to M$ (i.e. can be extended infinitely in both directions).
Equivalently: if exp_p is defined on T_pM for all $p \in M$.

Theorem 5.26. (Hopf-Rinow) Let (M, g) be a connected Riemannian manifold with metric g. Then the following conditions are equivalent:

(a) (M, g) is complete (i.e. every Cauchy sequence converges);

(b) every closed and bounded subset is compact

(c) (M, g) is geodesically complete.

Moreover, every of the conditions above imply

(d) for every $p, q \in M$ there exists a minimal geodesic connecting p and q.

Remark. Theorem 5.24 uses the following notions defined in a metric space:

- $\{x_i\}$, $x_i \in M$ is a Cauchy sequence if $\forall \varepsilon\ \exists N : \forall m,n > N \quad d(x_m, x_n) < \varepsilon$;
- a set $A \subset M$ is <u>bounded</u> if $A \subset B_r(p)$ for some $r > 0$, $p \in M$;
- a set $A \subset M$ is <u>closed</u> if $\{x_n \in A, x_n \to x\} \Rightarrow x \in A$;
- a set $A \subset M$ is <u>compact</u> if each open cover has a finite subcover;
- a set $A \subset M$ is <u>sequentially compact</u> if each sequence has a converging subsequence.

Some properties: 1. A compact set is sequentially compact, bounded, closed.
2. A compact metric space is complete.
3. In a complete metric space, a sequentially compact set is compact.

6 Integration on Riemannian manifolds

Definition 6.1 Let (M,g) be a Riemannian manifold and $f : M \to \mathbf{R}$ be a function with $supp(f) \subset U$, where $\varphi : U \to V$ is a coordinate chart, $\varphi = (x_1, \dots, x_n)$. Then define

$$\int_M f = \int_M f d\ Vol = \int_U f d\ Vol = \int_V f \circ \varphi^{-1}(x) \sqrt{det(g_{ij}) \circ \varphi^{-1}(x)} dx,$$

where $g_{ij}(p) = \langle \frac{\partial}{\partial x_i} |_p, \frac{\partial}{\partial x_j} |_p \rangle$ for all $p \in U$.

Proposition 6.2. Definition 6.1 does not depend on the choice of coordinates.

Definition 6.3. A <u>volume</u> of a subset $A \subset U \subset M$ is defined by

$$VolA = \int_M 1_A\ d\ Vol = \int_A d\ Vol = \int_{\varphi(A)} \sqrt{det(g_{ij}) \circ \varphi^{-1}(x)} dx,$$

where $1_A : M \to \{0,1\}$, $1_A(p) = 1$ if $p \in A$ and 0 otherwise.

Example 6.4. Integration on $\mathbf{H}^2$.

Definition 6.5. A <u>partition of unity</u> is a set of smooth functions $\varphi_\alpha : M \to [0,1]$ such that $\sum_\alpha \varphi_\alpha(p) = 1$ $\forall p \in M$ and for every $p \in M$ there exists an open set U_p, $p \in U_p$ such that for all but finitely many of α holds $\varphi_\alpha|_{U_p} \equiv 0$.

Definition 6.6. Given an open cover $\{U_\alpha\}$ of M, the set of functions $\{\varphi_\alpha\}$ <u>subordinates</u> to $\{U_\alpha\}$ if $\overline{supp\ \varphi_\alpha} \subset U_\alpha$ for all α.

Fact 6.7. For any countable atlas U_α there exists a partition of unity which subordinates to $\{U_\alpha\}$.

Corollary 6.8. For a Riemannian manifold M with countable atlas and subordinate partition of unity $\{\varphi_\alpha\}$ one has $\int_M f d\ Vol = \sum_\alpha \int_{U_\alpha} f \cdot \varphi_\alpha d\ Vol$.

Remark 6.9. In practice, one chooses (if possible) a chart $U \subset M$ such that $Vol(M \setminus U) = 0$, then $\int_M f\ d\ Vol = \int_U f\ d\ Vol$.

Remark 6.10. Isometries preserve the volume,
i.e. if $\psi : (M,g) \to (N,h)$ is an isometry then $\int_N f\ d\ Vol = \int_M f \circ \psi\ d\ Vol$.