Riemannian Geometry, Michaelmas 2013.

Smooth manifolds and Tangent space: outline

1 Smooth manifolds

"Smooth" means "infinitely differentiable", C^∞.

Definition 1.1. Let M be a set. An n-dimensional smooth atlas on M is a collection of triples $(U_\alpha, V_\alpha, \varphi_\alpha)$, where $\alpha \in I$ for some indexing set I, s.t.

0. $U_\alpha \subseteq M$; $V_\alpha \subseteq \mathbf{R}^n$ is open;
1. $\bigcup_{\alpha \in I} U_\alpha = M$;
2. Each $\varphi_\alpha : U_\alpha \to V_\alpha$ is a bijection;
3. The composition $\varphi_\beta \circ \varphi_\alpha^{-1} \mid_{\varphi_\alpha(U_\alpha \cap_\beta)}$: $\varphi_\alpha(U_\alpha \cap_\beta) \to \varphi_\beta(U_\alpha \cap_\beta)$ is a smooth map for all ordered pairs (α, β), where $\alpha, \beta \in I$.

The number n is called the dimension of M, the maps φ_α are called coordinate charts, the compositions $\varphi_\beta \circ \varphi_\alpha^{-1}$ are called transition maps or change of coordinates.

Definition 1.2. M is called a smooth n-dimensional manifold if

1. M has an n-dimensional smooth atlas;
2. M is Hausdorff (see Def. 1.4 below);
3. M is second-countable (technical condition, we will ignore).

Definition 1.3. Let M have a smooth atlas. We call a set $A \subseteq M$ open iff for each $\alpha \in I$ the set $\varphi_\alpha(A \cap U_\alpha)$ is open in $\mathbf{R}^n$. This defines a topology on M.

Definition 1.4. M is called Hausdorff if for each $x, y \in M$, $x \neq y$ there exist open sets A_x and A_y such that $x \in A_x$, $y \in A_y$ and $A_x \cap A_y = \emptyset$.

Example 1.6. Examples of smooth manifolds: sphere, torus, Klein bottle, 3-torus, real projective space.

Definition 1.7. Let $f : M^m \to N^n$ be a map of smooth manifolds with atlases $(U_i, \varphi_i(U_i), \varphi_i)_{i \in I}$ and $(U_j, \psi_j(U_j), \psi_j)_{j \in J}$. The map f is smooth if it induces smooth maps between the open sets in $\mathbf{R}^m$ and $\mathbf{R}^n$, i.e. if $\psi_j \circ f \circ \varphi_i^{-1} \mid_{\varphi_i(f^{-1}(V_j \cap f(U_i)))}$ is smooth for all $i \in I$, $j \in J$.
If f is a bijection and both f and f^{-1} are smooth then f is called a diffeomorphism.

Definition 1.8. Let $U \subseteq \mathbf{R}^n$ be open, $m < n$ and $f : U \to \mathbf{R}^m$ be a smooth map. Let $Df \mid_x = (\frac{\partial f_i}{\partial x_j})$ be the matrix of partials at $x \in U$ (differential). Then

- $x \in \mathbf{R}^n$ is a regular point of f if $rk(Df \mid_x) = m$;
- $y \in \mathbf{R}^m$ is a regular value of f if $f^{-1}(\{y\})$ consists of regular points only.

Theorem 1.9. (Implicit Function Theorem).
If $y \in f(U)$ is a regular value of f then $f^{-1}(y)$ is an $(n-m)$-dimensional smooth manifold.

Examples 1.10-1.11. An ellipsoid is a smooth manifold. Matrix groups are smooth manifolds.

Definition 1.12. A Lie group is a smooth manifold G together with a group operation $G \times G \to G$ s.t. the maps $(g_1, g_2) \to g_1 \cdot g_2$ and $g \to g^{-1}$ are smooth.
In particular, all matrix groups are Lie groups.

2 Tangent space

Definition 2.1. Let M be a smooth manifold, $p \in M$. Then $C^\infty(M,p)$ is a set of all smooth functions on M defined in a neighbourhood of p.

Definition 2.2. A derivation on $C^\infty(M,p)$ is a linear map $\delta : C^\infty(M,p) \to \mathbf{R}$, s.t. for all $f, g \in C^\infty(M,p)$ holds $\quad \delta(f \cdot g) = f(p)\delta(g) + \delta(f)g(p) \quad$ (the Leibniz rule).
Denote by $\mathcal{D}^\infty(M,p)$ the set of all derivations. Check that it is a real vector space.

Definition 2.3. The set $\mathcal{D}^\infty(M,p)$ is called the tangent space of M at p, denoted T_pM.

Definition 2.4. Let $\gamma : (a,b) \to M$ be a smooth curve in M, $\gamma(t_0) = p$ and $f \in C^\infty(M,p)$. Define the directional derivative of f at p along γ by $\gamma'(t_0)(f) \in \mathbf{R}$:

$$\gamma'(t_0)(f) = \lim_{s\to 0} \frac{f(\gamma(t_0+s)) - f(\gamma(t_0))}{s} = (f \circ \gamma)'(t_0) = \frac{d}{dt}\Big|_{t=t_0}(f \circ \gamma)$$

Check that the directional derivatives satisfy the properties of derivations.

Remark. Two curves γ_1 and γ_2 through p define the same directional derivative iff they have the same direction and the same speed at p.

Notation. Let M^n be a manifold, $\varphi : U \to V \subseteq \mathbf{R}^n$ a chart at $p \in U \subset M$. For $i = 1, \dots, n$ define the curves $\gamma_i(t) = \varphi^{-1}(\varphi(p) + e_i t)$ for small $t > 0$ (here e_i is a basis of $\mathbf{R}^n$).

Def. 2.5. $\frac{\partial}{\partial x_i}\big|_p := \gamma_i'(0)$, i.e.

$$\frac{\partial}{\partial x_i}\Big|_p(f) = (f \circ \gamma_i)'(0) = \frac{d}{dt}(f \circ \varphi^{-1})(\varphi(p) + te_i)\Big|_{t=0} = \frac{\partial}{\partial x_i}(f \circ \varphi^{-1})(\varphi(p))$$

($\frac{\partial}{\partial x_i}$ on the right is just a classical partial derivative).

Proposition 2.6. $\langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \rangle = \{\text{Directional Derivatives}\} = \mathcal{D}^\infty(M,p)$.

Lemma 2.7. Let $\varphi : U \subseteq M \to \mathbf{R}^n$ be a chart, $\varphi(p) = 0$. Let $\tilde{\gamma}(t) = \sum_{i=1}^n (k_i t e_i) \in \mathbf{R}^n$ be a curve (straight ray), where $\langle e_1, \dots, e_n \rangle$ is a basis. Let $\gamma(t) = \varphi^{-1}(t) \in M$, $p \in \gamma(0)$. Then $\gamma'(0) = \sum_{i=1}^n k_i \frac{\partial}{\partial x_i}$.

Example 2.8. For the group $SL(n,\mathbf{R}) = \{A \in M_n \mid detA = 1\}$, the tangent space at I is the set of all trace-free matrices: $T_I(SL(n,\mathbf{R})) = \{X \in M_n(\mathbf{R}) \mid trX = 0\}$.

Proposition 2.9. (Change of basis for T_pM). Let M^n be a smooth manifold, $\varphi_\alpha : U_\alpha \to V_\alpha$ a chart, $(x_1^\alpha, \dots, x_n^\alpha)$ the coordinates in V_α. Then $\frac{\partial}{\partial x_j^\alpha}\mid_p = \sum_{i=1}^n \frac{\partial x_i^\beta}{\partial x_j^\alpha}\frac{\partial}{\partial x_i^\alpha}$.

Definition 2.10. Let M, N be smooth manifolds, let $f : M \to N$ be a smooth map. Define a linear map $Df(p) : T_pM \to T_{f(p)N}$ called the differential of f at p by $Df(p)\gamma'(0) = (f \circ \gamma)'(0)$ for a smooth curve $\gamma \in M$ with $\gamma(0) = p$.

Lemma 2.11. (a) $D(id)(p) : T_pM \xrightarrow{I} T_pM$;
(b) for $M \xrightarrow{g} N \xrightarrow{f} L$ holds $D(g \circ f)\big|_p = Dg\big|_{f(p)} \circ Df\big|_p$.

Tangent bundle and vector fields

Definition 2.13. Let M be a smooth manifold. A disjoint union $TM = \cup_{p\in M} T_pM$ of tangent spaces to each $p \in M$ is called a tangent bundle.
There is a map $\Pi : TM \to M$ (called projection), $\Pi(v) = p$ if $v \in T_pM$.

Proposition 2.14. The tangent bundle TM if M^n has a structure of $2n$-dimensional smooth manifold and $\Pi : TM \to M$ is a smooth map.

Definition 2.15. A vector field X is a "section" of the tangent bundle, that is a smooth map $X : M \to TM$ such that $\Pi \circ X = id_M$ is an identity map on M.
The set of all vector fields on M is denoted $\mathfrak{X}(M)$. This set has a structure of a vector space.

Remark 2.16. Taking a coordinate chart $(U, \varphi = (x_1, \dots, x_n)))$ we can write any vector field $X\mid_U$ as $X(p) = \sum_{i=1}^n f_i(p) \frac{\partial}{\partial x_i} \in T_pM$.

Examples 2.17-2.18: vector fields on the torus and 3-sphere.

Remark 2.19. Since for $X \in \mathfrak{X}(M)$ we have $X_p \in T_pM$ which is a directional derivative at $p \in M$, we can use the vector field to differentiate a function $f \in C^\infty(M)$, $f : M \to \mathbf{R}$ by $(Xf)(p) = X(p)f = \sum a_i(p) \frac{\partial f}{\partial x_i}\Big|_p$, so that we get another smooth function $X_f \in C^\infty(M)$.

Proposition 2.20. Let $X, Y \in \mathfrak{X}(M)$. Then there exists a unique vector field $Z \in \mathfrak{X}(M)$ such that $Z(f) = X(Y(f)) - Y(X(f))$ for all $f \in C^\infty(M)$.

This vector field $Z(f) = X(Y(f)) - Y(X(f))$ is denoted $[X, Y]$ and called the Lie bracket.

Proposition 2.21. Properties of the Lie bracket:

a. $[X, Y] = -[Y, X]$;

b. $[aX + bY, Z] = a[X, Z] + b[Y, Z]$ for $a, b \in \mathbf{R}$;

c. $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ (Jacobi identity);

d. $[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$ for $f, g \in C^\infty(M)$.

Definition 2.22. A Lie algebra is a vector space $\mathfrak{g}$ with a binary operation $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the Lie bracket which satisfies properties a,b,c of Proposition 2.21.
In other words, $\mathfrak{X}(M)$ is a Lie algebra.

Theorem 2.23. (The Hairy Ball Theorem).
There is no non-vanishing continuous vector field on an even-dimensional sphere S^{2m}.