Riemannian Geometry, Michaelmas 2013.

# Riemannian metric, Levi-Civita connection and parallel transport: outline

## 3 Riemannian metric

**Definition 3.1.** Let $M$ be a smooth manifold. A Riemannian metric written $g_p(\cdot,\cdot)$ or $\langle\cdot,\cdot\rangle_p$ is a family of real inner products $g_p : T_pM \times T_pM \to \mathbf{R}$ depending smoothly on $p \in M$.
A smooth manifold $M$ with a Reimannian metric $g$ is called a Riemannian manifold $(M, g)$.

**Examples 3.2–3.3.** Euclidean metric in $\mathbf{R}^n$, induced metric on $M \in \mathbf{R}^n$.

**Definition 3.4.** Let $(M, g)$ be a Riemannian manifold. For $v \in T_pM$ define the length of $v$ by $0 \le ||v||_g = \sqrt{g_p(v, v)}$.
Suppose $c : [a, b] \to M$ is a smooth curve on $M$. Define the length of $c$ by $L(c) = \int_a^b ||c'(t)||dt$.
(this does not depend on parametrization, see Theorem 3.10).

**Remark 3.5.** Let $M \in \mathbf{R}^n$ be a smooth manifold given by $f(x_1, \dots, x_n) = a$. Let $p \in M$, $v \in T_pM$. Then $v$ satisfies $\sum_{i=1}^n \frac{\partial f}{\partial x_i} v_i = 0$.

**Example 3.6.** three models of hyperbolic geometry:

| model | notation | $M$ | $g$ |
|---|---|---|---|
| Hyperboloid | $\mathbf{W}^n$ | $\{y \in \mathbf{R}^{n+1} \mid q(y,y) = -1, y_{n+1} > 0\}$ where $q(x,y) = \sum_{i=1}^n x_i y_i - x_{n+1}y_{n+1}$ | $\langle v, w\rangle = q(v, w)$ |
| Poincaré ball | $\mathbf{B}^n$ | $\{x \in \mathbf{R}^n \mid \|\|x\|\|^2 = \sum\limits_{i=1}^n x_i^2 < 1\}$ | $g_x(v,w) = \frac{4}{(1-\|\|x\|\|^2)^2}\langle v, w\rangle$ |
| Upper half-space | $\mathbf{H}^n$ | $\{x \in \mathbf{R}^n \mid x_n > 1\}$ | $g_x(v,w) = \frac{1}{x_n^2}\langle v, w\rangle$ |

**Definition 3.7.** Given two vector spaces $V_1, V_2$ with real inner products $(V_i, \langle\cdot,\cdot\rangle_i)$ we call a linear isomorphism $T : V_1 \to V_2$ a linear isometry if $\langle Tv, Tw\rangle_2 = \langle v, w\rangle_1$ for all $v, w \in V_1$.
This is equivalent to preservation of the lengths of all vectors since $\langle v, w\rangle = \frac{1}{2}(\langle v+w, v+w\rangle - \langle v, v\rangle - \langle w, w\rangle)$.
A diffeomorphism $f : (M, g) \to (N, h)$ of two Riemannian manifolds is an isometry if $DF(p) : T_pM \to T_{f(p)N}$ is a linear isometry for all $p \in M$.

**Example 3.9.** Isometry $f : \mathbf{H}^2 \to \mathbf{B}^2$ given by $f(z) = \frac{z-i}{z+i}$.

**Theorem 3.10.** (Reparametrization). Let $\varphi : [c, d] \to [a, b]$ be a strictly monotonic smooth function, $\gamma : [a, b] \to M$ a smooth curve. Then for $\tilde{c} = c \circ \varphi : [c, d] \to M$ (reparametrization of $c$) holds $L(c) = L(\tilde{c})$.

**Definition 3.11.** A differentiable curve $c : [a, b] \to M$ is called an arc-length parametrization if $||c'(t)|| = 1$. Every curve has an arc-length parametrization $\tilde{c}(t) : [0, L(c|_{[a,b]})] \to M$, $\tilde{c}(t) = c \circ \varphi(t)$, where $\varphi^{-1}(t) = L(c|_{[a,t]})$.

**Example 3.12.** Length of vertical segments in $\mathbf{H}$. Vertical half-lines are geodesics.

**Definition 3.13.** Define a distance $d : M \times M \to [0, \infty)$ by $d(p, q) = inf_\gamma\{L(\gamma)\}$, where $\gamma$ is a smooth curve with end $p$ and $q$.
A curve $c(t) : [a, b] \to M$ is geodesic if $d(c(x), c(y)) = L(c|_{[x,y]})$ for all $x, y \in [a, b]$ $(x < y)$.

**Remark 3.14.** $d$ turns $(M, g)$ into a metric space.

**Definition 3.15.** If $(X, d)$ is a metric space then any subset $A \in X$ is also a metric space with the induced metric $d|_{A\times A} : A \times S \to [0, \infty)$.

**Example 3.16** Punctured Riemannian sphere: $\mathbf{R}^n$ with $g_x(v, w) = \frac{4}{(1+||x||^2)^2}\langle v, w\rangle$.

# 4 Levi-Civita connection and parallel transport

## 4.1 Levi-Civita connection

**Example 4.1** In $\mathbf{R}^n$, given a vector field $X = \sum a_i(p)\frac{\partial}{\partial x_i} \in \mathfrak{X}(\mathbf{R}^n)$ and a vector $v \in T_p\mathbf{R}^n$ define the covariant derivative of $X$ in direction $v$ by $\nabla_v(X) = \lim\limits_{t\to 0} \frac{X(p+tv)-X(p)}{t} = \sum v(a_i)\frac{\partial}{\partial x_i}\big|_p \in T_p\mathbf{R}^n$.

**Properties 4.2.** In $\mathbf{R}^n$, the covariant derivative $\nabla_v X$ satisfies properties (a)-(e) listed below in Definition 4.3 and Theorem 4.4.

**Definition 4.3.** Let $M$ be a smooth manifold. A map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, $X, Y \nabla_X Y$ is called a covariant derivative or affine connection if for all $X, Y, Z \in \mathfrak{X}(M)$ and $f, g \in C^\infty(M)$ holds

(a) $\nabla_X(Y + Z) = \nabla_X(Y) + \nabla_X(Z)$

(b) $\nabla_X(fY) = X(f)Y(p) + f(p)\nabla_X Y$

(c) $\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z$

**Theorem 4.4.** (Levi-Civita, Fundamental Theorem of Riemannian Geometry). Let $(M, g)$ be a Riemannian manifold. There exists a unique covariant derivative $\nabla$ on $M$ with the additional properties for all $X, Y, Z \in \mathfrak{X}(M)$:

(d) $v(\langle X, Y\rangle) = \langle \nabla_v X, Y\rangle + \langle X, \nabla_v Y\rangle$ (Riemannian property);

(e) $\nabla_X Y - \nabla_y X = [X, Y]$ (Torsion-free)

This connection is called Levi-Civita connection of $(M, g)$.

**Example 4.5.** Levi-Civita connection in $\mathbf{R}^n$ and in $M \subset \mathbf{R}^n$ with induced metric.

## 4.2 Christoffel symbols

**Definition 4.6.** Let $\nabla$ be a Levi-Civita connection on $(M, g)$ and $\varphi : U \to V$ a coordinate chart with coordinates $\varphi = (x_1, \ldots, x_n)$. Then we have $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}(p) \in T_pM$. i.e. there exist uniquely determined $\Gamma^k_{ij} \in C^\infty(U)$ with $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}(p) = \sum_{k=1}^n \Gamma^k_{ij} \frac{\partial}{\partial x_k}(p)$. These functions are called Christoffel symbols of $\nabla$ with respect to the chart $\varphi$.

They characterize $\nabla$ since $$\nabla_{\sum\limits_{i=1}^n a_i \frac{\partial}{\partial x_i}} \sum_{j=1}^n b_j \frac{\partial}{\partial x_j} = \sum_{i,j} a_i \frac{\partial b_j}{\partial x_i}\frac{\partial}{\partial x_j} + \sum_{i,j,k} a_i b_j \Gamma^k_{i,j}\frac{\partial}{\partial x_k}.$$

**Proposition 4.7.** $\Gamma^k_{ij} = \frac{1}{2}\sum\limits_k g^{ks}(g_{ik,j} + g_{jk,i} - g_{ij,k})$, where $g_{ab,c} = \frac{\partial}{\partial x_c} g_{ab}$ and $(g^{ij}) = (g_{ij})^{-1}$.

In particular, $\Gamma^k_{ij} = \Gamma^k_{ji}$.

**Example 4.8.** In $\mathbf{R}^n$, $\Gamma^k_{ij} \equiv 0$ for all $i, j, k$; $\Gamma^k_{ij}$ in $S^2 \subset \mathbf{R}^3$ with induced metric.

## 4.3 Parallel transport

**Definition 4.9.** Let $c : (a, b) \to M$ be a differentiable curve. A map $X : (a, b) \to TM$ with $X(t) \in T_{c(t)}M$ is called a vector field along $c$. Denote the space of all these maps by $\mathfrak{X}_c(M)$.

**Example 4.10.** $c'(t) \in \mathfrak{X}_c(M)$.

**Proposition 4.11.** Let $(M,g)$ be a Riemannian manifold, $\nabla$ be a Levi-Civita connection, $c:(a,b)\to M$ be a differentiable curve. There exists a unique map $\frac{D}{\partial t}:\mathfrak{X}_c(M)\to\mathfrak{X}_c(M)$ satisfying

(a) $\frac{D}{\partial t}(X+Y)=\frac{D}{\partial t}X+\frac{D}{\partial t}Y$

(b) $\frac{D}{\partial t}(fX)=f'(t)X+f\frac{D}{\partial t}X$, for all differentiable $f:(a,b)\to\mathbf{R}$

(c) If $\widetilde{X}\in\mathfrak{X}(M)$ is a local extension of $X$
(i.e. there exists $t_0\in(a,b)$ and $\varepsilon>0$ such that $X(t)=\widetilde{X}\big|_{c(t)}$ for all $t\in(t_0-\varepsilon,t_0+\varepsilon)$)
then $(\frac{D}{\partial t}X)(t_0)=\nabla_{c'(t_0)}\widetilde{X}$.

This map $\frac{D}{\partial t}:\mathfrak{X}_c(M)\to\mathfrak{X}_c(M)$ is called the covariant derivative along the curve $c$.

**Example 4.12.** For a surface $M\in\mathbf{R}^3$ the condition $\frac{D}{\partial t}(c'(t))\equiv 0$ is equivalent to $c''(t)\perp T_{c(t)}M$, which is in its turn the condition for $c$ to be geodesic known from the course of Differential Geometry.

**Definition 4.13.** Let $X\in\mathfrak{X}_c(M)$. If $\frac{D}{\partial t}X=0$ then $X$ is said to be parallel along $c$.

**Example 4.14.** In $\mathbf{R}^n$ it means that $X$ does not depend on the point $p\in\mathbf{R}^n$.

**Theorem 4.15.** Let $c:[a,b]\to M$ be a smooth curve, $v\in T_{c(a)M}$. Then there exists a unique vector field $X\in\mathfrak{X}_c(M)$ with $X(a)=v\in T_{c(a)}M$.

**Example 4.16.** Parallel vector fields form a vector space of dimension $n$ (for $n$-dimensional $(M,g)$).

**Definition 4.17.** Let $c:[a,b]\to M$ be a smooth curve, A linear map $P_c:T_{c(a)}M\to T_{c(b)}M$ called parallel transport defined by $P_c(v)=X(b)$ where $X\in\mathfrak{X}_c(M)$ with $X(a)=v$, $\frac{D}{\partial t}X=0$.

**Remark.** (a) The parallel transport $P_c$ depend on the curve $c$ (not only on its endpoints).
(b) The parallel transport is a linear isometry $P_c:T_{c(a)}M\to T_{c(b)}M$, i.e. $g_{c(a)}(v,w)=g_{c(b)}(P_cv,P_cw)$.