Riemannian Geometry, Michaelmas 2013.

Geodesics

5.1 Geodesics as solutions to ODEs

Definition 5.1. Given (M,g), the curve $c:[a,b]\to M$ is a <u>geodesic</u> if $\frac{D}{dt}c'(t)=0$ for all $t\in[a,b]$.

Lemma 5.2. If c is a geodesic that c is parametrized proportional to the arc length.

Theorem 5.3. Given a Riemannian manifold (M,g), $p\in M$, $v\in T_pM$, there <u>exists</u> $\varepsilon>0$ and a <u>unique</u> geodesic $c:(-\varepsilon,\varepsilon)\to M$ such that $c(0)=p$, $c'(0)=v$.

Example 5.4–5.5 Geodesics in Euclidean space, on a sphere, and in the upper half-plane model $\mathbf{H}^2$.

Remark. Differential equations for geodesics: $c_k''(t)=-\sum_{ij}c_i(t)c_j(t)\Gamma_{ij}^k(c(t)),\quad k=1,\dots,n.$

5.2 Geodesics as distance-minimizing curves. First variation formula of the length.

Definition 5.6. Let $c:[a,b]\to M$ be a smooth curve. A smooth map $F:(-\varepsilon,\varepsilon)\times[a,b]\to M$ is a differentiable <u>variation of c</u> if $F(0,t)=c(t)$.
Variation is <u>proper</u> if $F(s,a)=c(a)$ and $F(s,b)=c(b)$ for all $s\in(-\varepsilon,\varepsilon)$.
Variation may be considered as a <u>family</u> of the curves $F_s(t)=F(s,t)$.

Definition 5.7. A <u>variation vector field</u> X of the variation F is defined by $X(t)=\frac{\partial F}{\partial s}(0,t)$.

Definition 5.8. The <u>length</u> and <u>energy</u> of variation are

$l(s):=\int\limits_a^b||\frac{\partial F}{\partial t}(s,t)||dt,\quad l:(-\varepsilon,\varepsilon)\to[0,\infty);\qquad E(s):=\int\limits_a^b||\frac{\partial F}{\partial t}(s,t)||^2dt,\quad E:(-\varepsilon,\varepsilon)\to[0,\infty).$

Remark: $l(s)$ is the length of the curve $F_s(t)$.

Theorem 5.9. A smooth curve c is geodesic if and only if $l'(0)=0$ for each proper variation and c is parametrized proportionally to the arc length.

Corollary 5.10. Let $c:[a,b]\to M$ be the shortest curve from $c(a)$ to $c(b)$, and c is parametrized proportionally to the arc length. Then c is geodesic.

Lemma 5.11. (Symmetry Lemma). Let $W\subset\mathbf{R}^2$ be an open set and $F:W\to M$, $(s,t)\mapsto F(s,t)$ be a differentiable map. Let $\frac{D}{dt}$ be the covariant derivative along $F_s(t)$ and $\frac{D}{ds}$ be the covariant derivative along $F_t(s)$. Then $\frac{D}{dt}\frac{\partial F}{\partial s}=\frac{D}{ds}\frac{\partial F}{\partial t}$.

Theorem 5.12. (First variation formula of the length). Let $F:(-\varepsilon,\varepsilon)\times[a,b]\to M$ be a variation of $c(t)$, $c'(t)\neq0$. Let $X(t)$ be its variation vector field and $l:(-\varepsilon,\varepsilon)\to[0,\infty)$ its length. Then

$$l'(0)=\int_a^b\frac{1}{||c'(t)||}\frac{d}{dt}\langle X(t),c'(t)\rangle dt-\int_a^b\frac{1}{||c'(t)||}\langle X(t),\frac{D}{dt}c'(t)\rangle dt.$$

Corollary 5.13.

- If in addition $c(t)$ is parametrized proportionally to the arc length, $||c'(t)||\equiv c$ then $l'(0)=\frac{1}{c}\langle X(b),c'(b)\rangle-\frac{1}{c}\langle X(a),c'(a)\rangle-\frac{1}{c}\int_a^b\langle X(t),\frac{D}{dt}c'(t)\rangle dt$;
- if $c(t)$ is geodesic, then $l'(0)=\frac{1}{c}\langle X(b),c'(b)\rangle-\frac{1}{c}\langle X(a),c'(a)\rangle$;
- if F is proper <u>and</u> c is parametrised proportionally to the arc length, then $l'(0)=-\frac{1}{c}\int_a^b\langle X(t),\frac{D}{dt}c'(t)\rangle dt$;
- if F is proper <u>and</u> c is geodesic, then $l'(0)=0$.

Lemma 5.14. Any vector field X along $c(t)$ with $X(a)=X(b)=0$ is a variation vector field for some proper variation F.

5.3 Exponential map and Gauss Lemma

Let $p \in M$, $v \in T_pM$. Denote by $c_v(t)$ the unique maximal (by inclusion) geodesic with $c_v(0) = p$, $c_v'(0) = v$.

Definition 5.15. If $c_v(1)$ exists, define $exp_p : T_pM \to M$ by $v \mapsto c_v(1)$, the exponential map at p.

Example 5.16. Exponential map on the sphere S^2: length of c_v from p to $c_v(1)$ equals to $||v||$.

Notation. $B_r(O_p) = \{v \in T_pM \mid ||v|| < r\} \subset T_pM$ is a ball of radius r centered at $p = O_p$.

Proposition 5.17. (without proof).
For any $p \in (M, g)$ there exists an $r > 0$ such that $exp_p : B_r(O_p) \to exp_p(B_r(O_p))$ is a diffeomorphism.

Example. On S^2 the set $exp_p(B_{\pi/2}(O_p))$ is a hemisphere, so that every geodesic starting from p is orthogonal to the boundary of this set.

Theorem 5.18. (Gauss Lemma). Let (M, g) be a Riemannian manifold, $p \in M$ and let $\varepsilon > 0$ be such that $exp_p : B_\varepsilon(O_p) \to exp_p(B_\varepsilon(O_p))$ is a diffeomorphism. Define a hypersurface $A_\delta := \{exp_p(w) \mid ||w|| = \delta\}$ for all $0 < \delta < \varepsilon$. Then every radial geodesic $c : t \to exp_p(tv)$, $t \geq 0$ is orthogonal to A_δ.

Remark 5.19. The curve $c_v(t) = exp_p(tv)$ is indeed a geodesic!

Lemma 5.19. Let $p \in M$ and let $\varepsilon > 0$ be such that $exp_p : B_\varepsilon(O_p) \to exp_p(B_\varepsilon(O_p))$ is a diffeomorphism. Take $\gamma : [0, 1] \to exp_p(B_\varepsilon(O_p))$. Then there exists a curve $v(s) : [0, 1] \to T_p(M)$, $||v(s)|| = 1$ and a non-negative function $r(s) : [0, 1] \to \mathbf{R}_+$ such that $\gamma(s) = exp_p(r(s) \cdot v(s))$.

Lemma 5.20. Let $r : [0, 1] \to \mathbf{R}$, $v : [0, 1] \to S_pM = w \in T_pM \mid ||w|| = 1\}$. Define $\gamma : [0, 1] \to exp_p(B_\varepsilon(O_p))$ by $\gamma(s) = exp_p(r(s)v(s))$. Then $l(\gamma) \geq |r(1) - r(0)|$ for the length $l(\gamma)$ of γ and the equality holds if and only if γ is a reparametrisation of a radial geodesic (i.e. if and only if $v(s) \equiv const = v(0)$, $r(s)$ is increasing or decreasing function).

Corollary 5.21. Given a point $p \in M$, there exists $\varepsilon > 0$ such that for any $q \in B_\varepsilon(O_p)$ there exists a curve $c(t)$ connecting p and q and satisfying $l(c) = d(p, q)$. (This curve is a radial geodesic).

Notation. Denote $B_\varepsilon(p) := exp_p(B_\varepsilon(O_p)) \subset M$, a geodesic ball and $S_\varepsilon(p) = \partial B_\varepsilon(p)$, a geodesic sphere. Note, $B_\varepsilon(p) = \{q \in M \mid d(p, q) \leq \varepsilon\}$.

Proposition 5.22. (without proof). Let $p \in M$. Then there exists an open set U_p, $p \in U$ and an ε such that for any $q \in U$ the map $exp_q : B_\varepsilon(O_q) \to B_\varepsilon(q)$ are diffeomorphisms.

Remark 5.23. (Naturality of exponential map).
Let $\varphi : (M, g) \to (N, h)$ be an isometry. Then $D\varphi = exp^{-1}_{\varphi(p)} \circ \varphi \circ exp_p$.

5.4 Hopf-Rinow Theorem

Definition 5.24. A geodesic $c : [a, b] \to M$ is minimal if $l(c) = d(c(a), c(b))$.
A geodesic $c : \mathbf{R} \to M$ is minimal if its restriction $c|_{[a,b]}$ is minimal for each segment $[a, b] \in \mathbf{R}$.

Example: no minimal geodesics on S^2, all geodesics are minimal in $\mathbf{E}^2$ and $\mathbf{H}^2$.

Definition 5.25. A Riemannian manifold (M, g) is geodesically complete if every geodesic $c : [a, b] \to M$ can be extended to a geodesic $\tilde{c} : \mathbf{R} \to M$ (i.e. can be extended infinitely in both directions).
Equivalently: if exp_p is defined on T_pM for all $p \in M$.

Theorem 5.26. (Hopf-Rinow) Let (M, g) be a connected Riemannian manifold with distance d. Then the following conditions are equivalent:

(a) (M, g) is complete (i.e. every Cauchy sequence converges);

(b) every closed and bounded subset is compact

(c) (M, g) is geodesically complete.

Moreover, every of the conditions above imply

(d) for every $p, q \in M$ there exists a minimal geodesic connecting p and q.

Remark. Theorem 5.24 uses the following notions defined in a metric space:

- $\{x_i\}$, $x_i \in M$ is a Cauchy sequence if $\forall \varepsilon \ \exists N : \forall m, n > N \quad d(x_m, x_n) < \varepsilon$;
- a set $A \subset M$ is bounded if $A \subset B_r(p)$ for some $r > 0$, $p \in M$;
- a set $A \subset M$ is closed if $\{x_n \in A, x_n \to x\} \Rightarrow x \in A$;
- a set $A \subset M$ is compact if each open cover has a finite subcover;
- a set $A \subset M$ is sequentially compact if each sequence has a converging subsequence.

Some properties: 1. A compact set is sequentially compact, bounded, closed.
2. A compact metric space is complete.
3. In a complete metric space, a sequentially compact set is compact.

6 Integration on Riemannian manifolds

Definition 6.1 Let (M, g) be a Riemannian manifold and $f : M \to \mathbf{R}$ be a function with $supp(f) \subset U$, where $\varphi : U \to V$ is a coordinate chart, $\varphi = (x_1, \dots, x_n)$. Then define

$$\int_M f = \int_M f d\ Vol = \int_U f d\ Vol = \int_V f \circ \varphi^{-1}(x) \sqrt{det(g_{ij}) \circ \varphi^{-1}(x)} dx,$$

where $g_{ij}(p) = \langle \frac{\partial}{\partial x_i} \mid_p, \frac{\partial}{\partial x_j} \mid_p \rangle$ for all $p \in U$.

Proposition 6.2. Definition 6.1 does not depend on the choice of coordinates.

Definition 6.3. A volume of a subset $A \subset U \subset M$ is defined by

$$VolA = \int_M 1_A \ d\ Vol = \int_A d\ Vol = \int_{\varphi(A)} \sqrt{det(g_{ij}) \circ \varphi^{-1}(x)} dx,$$

where $1_A : M \to \{0, 1\}$, $1_A(p) = 1$ if $p \in A$ and 0 otherwise.

Example 6.4. Integration on $\mathbf{H}^2$.

Definition 6.5. A partition of unity is a set of smooth functions $\varphi_\alpha : M \to [0, 1]$ such that $\sum_\alpha \varphi_\alpha(p) = 1$ $\forall p \in M$ and for every $p \in M$ there exists an open set U_p, $p \in U_p$ such that for all but finitely many of α holds $\varphi_\alpha|_{U_p} \equiv 0$.

Definition 6.6. Given an open cover $\{U_\alpha\}$ of M, the set of functions $\{\varphi_\alpha\}$ subordinates to $\{U_\alpha\}$ if $\overline{supp\ \varphi_\alpha} \subset U_\alpha$ for all α.

Fact 6.7. For any countable atlas U_α there exists a partition of unity which subordinates to $\{U_\alpha\}$.

Corollary 6.8. For a Riemannian manifold M with countable atlas and subordinate partition of unity $\{\varphi_\alpha\}$ one has $\int_M f d\ Vol = \sum_\alpha \int_{U_\alpha} f \cdot \varphi_\alpha d\ Vol$.

Remark 6.9. In practice, one chooses (if possible) a chart $U \subset M$ such that $Vol(M \setminus U) = 0$, then $\int_M f\ d\ Vol = \int_U f\ d\ Vol$.

Remark 6.10. Isometries preserve the volume,
i.e. if $\psi : (M, g) \to (N, h)$ is an isometry then $\int_N f\ d\ Vol = \int_M f \circ \psi\ d\ Vol$.