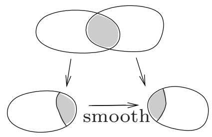
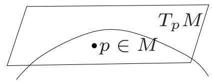
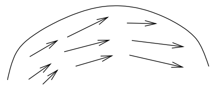
Riemannian Geometry, Epiphany 2014

Term 1: overview

Notion	Natation	What's that?	Picture	In coordinates
Smooth manifold	M	Smooth atlas Hausdorffness		$(x_1, \dots, x_n)$
Tangent Space	T_pM	Linear space of directional derivatives		$\langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \rangle$
Tangent bundle	TM	$TM = \bigcup_{p \in M} T_pM$		
Vector field Lie bracket	X $\mathfrak{X}(M)$ $[X, Y]$	Choice of vector in each T_pM (depends on p smoothly) All vector fields on M $[X,Y](f) = XY(f) - YX(f)$		$X(p) = \sum_{i=1}^n f_i(p) \frac{\partial}{\partial x_i}\Big\vert_p$
Riemannian metric	$g_p(\cdot,\cdot)$ or $< \cdot, \cdot >_p$	Inner product on each T_pM smoothly depending on p $\lvert\lvert v \rvert\rvert = \sqrt{g_p(v,v)}$, for $v \in T_pM$ $l(c) = \int_a^b \lvert\lvert c'(t) \rvert\rvert dt$, for a curve $c : [a,b] \to M$		$n \times n$ matrix (g_{ij}) $g_{ij}(p) = g_p(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$
Affine connection	$\nabla_X Y$	A map $\mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, $(X,Y) \mapsto \nabla_X Y$ linear by X, Y and satisfying Leibniz rule: $\nabla_X(fY) = X(f)Y(p) + f(p)\nabla_X Y$		
Levi-Civita connection		Torsion-free Riemannian affine connection, i.e. affine connection satisfying: $\nabla_X Y - \nabla_Y X = [X,Y]$ and $Z(\langle X, Y \rangle) = \langle \nabla_Z X, Y \rangle + \langle X, \nabla_Z Y \rangle$		$\nabla_{\sum_{i=1}^n a_i \frac{\partial}{\partial x_i}} (\sum_{j=1}^n b_j \frac{\partial}{\partial x_j}) =$ $= \sum_{i,j} a_i \frac{\partial b_j}{\partial x_i} \frac{\partial}{\partial x_j} + \sum_{i,j,k} a_i b_j \Gamma_{ij}^k \frac{\partial}{\partial x_k}$, $\Gamma_{ij}^k = \frac{1}{2} \sum_s g^{ks}(g_{ik,j} + g_{jk,i} - g_{ij,k})$, $g_{ab,c} = \frac{\partial}{\partial x_c} g_{ab}$, $(g^{ij}) = (g_{ij})^{-1}$
Covariant derivative along c	$\frac{D}{\partial t}$	Linear map $\frac{D}{dt} : \mathfrak{X}_c(M) \to \mathfrak{X}_c(M)$ satisfying Leibniz rule and $\frac{D}{dt} X(t_0) = \nabla_{c'(t_0)} \widetilde{X}$ where $X \in \mathfrak{X}_c(M)$ and $\widetilde{X}$ is a local extension of X.		
Geodesics		A curve $c(t)$ s.t. $\frac{D}{dt} c'(t) = 0$ along $c(t)$		$c_k'' = -\sum_{i,j} c_i'(t) c_j'(t) \Gamma_{ij}^k (c(t))$
Exponential map	exp_p	$exp_p : T_pM \to M$ $exp_p : v \mapsto c_v(1)$ where $c_v(t)$ is geodesic s.t. $c_v'(0) = v \in T_pM$