Riemannian Geometry, Michaelmas 2013.

Typical problems

Smooth manifolds, tangent spaces, vector fields

1. Smooth manifolds:
   - given M, construct an atlas;
   - given M, prove that it is a manifold (use Implicit Function Theorem or construct an atlas).
2. Tangent spaces:
   - find T_pM for some $p \in M$;
   - express some $X \in T_pM$ through $\frac{\partial}{\partial x_i}$ in some chart;
   - find the curves $\gamma_1, \dots, \gamma_n \in M$ s.t. $(\gamma_1'(0), \dots, \gamma_n'(0))$ form a basis of T_pM;
   - write $Df : T_p(M) \to T_{f(p)}N)$ for $f : M \to N$;
   - find $dim(ker(Df))$ or $ker(Df)$.
3. Vector fields:
   - given $X = \sum f_i \frac{\partial}{\partial x_i}$, $Y = \sum g_j \frac{\partial}{\partial x_j}$ compute $[X, Y]$;
   - show that some vector field in $\mathbf{R}^n$ restricts to a vector field on some manifold $M \subset \mathbf{R}^n$.

Riemannian metric, Levi-Civita connection and parallel transport

1. Riemannian metric:
   - given M and the bilinear form $g(v, w)$ show that g may be used to define a metric on M; — g is bilinear, positive definite, symmetric; smooth
   - find the length $L(c)$ for a curve $c \in M$; — $L(c) = \int_a^b ||c'(t)||dt$, $||v||_g = \sqrt{g_p(v,v)}$
   - for $N \subset \mathbf{R}^n$ find the induced mentric on N (in some chart);
   - given a map $f : (M, g) \to (N, h)$ check that it is an isometry; — $\langle Tv, Tw \rangle_h = \langle v, w \rangle_g$
   - same for a map $f : (M, g) \to (M, g)$;
   - for an isometry f find Df;
   - for a curve $c(t) \subset M$ find its arc-length parametrization. — $\tilde{c}(t) = c \circ \varphi(t)$, where $\varphi^{-1} = L(c|_{[a,t]})$
2. Levi-Civita connection and Christoffel symbols:
   - find Γ_{ij}^k in some chart on (M, g); — $\Gamma_{ij}^s = \frac{1}{2}\sum_k g^{sk}(g_{jk,i} + g_{ik,j} - g_{ij,k})$
   - find $\nabla_v Y$ given $v \in T_pM$, $Y \in \mathfrak{X}(M)$ on (M, g). — $\nabla_{\sum\limits_{i=1}^n a_i \frac{\partial}{\partial x_i}} \sum\limits_{j=1}^n b_j \frac{\partial}{\partial x_j} = \sum\limits_{i,j} a_i \frac{\partial b_j}{\partial x_i}\frac{\partial}{\partial x_j} + \sum\limits_{i,j,k} a_i b_j \Gamma_{i,j}^k \frac{\partial}{\partial x_k}$
3. Parallel transport:
   - In (M, g), find a curve $c(t)$, $c(0) = p$, $c'(0) = v$, $\frac{D}{\partial t}c'(t) = 0$ (i.e. find a parallel vector field on $c(t)$); — $\frac{D}{dt}(\sum a_i(t)\frac{\partial}{\partial x_i}\mid_{c(t)}) = \sum_i (a_i'\frac{\partial}{\partial x_i} + a_i \nabla_{c'(t)}\frac{\partial}{\partial x_i})$
   - find $\frac{D}{\partial t}X$ for $X = \sum a(t)\frac{\partial}{\partial x_i}\mid_{c(t)}$. When X is parallel? — $\frac{D}{\partial t}X = 0 \;\Leftrightarrow\; a_k' + \sum\limits_{i,j} a_i c_j' \Gamma_{ij}^k = 0 \qquad (k = 1, \dots, n)$
   - find the parallel transport $P_c : T_{c(a)}M \to T_{c(b)}M$; — Problem class 2