Durham University, Anna Felikson Michaelmas 2015

Riemannian Geometry IV, Homework 1 (Week 1)

Due date for starred problems: **Thursday, October 29**.

1.1. ($\star$) Let M be a smooth manifold of dimension m and N be a smooth manifold of dimension n. Show that the cartesian product

$$M \times N := \{(x, y) \mid x \in M, y \in N\}$$

is a smooth manifold of dimension $m + n$.

1.2. Consider the *Lemniscate of Gerono* Γ, which is given as a subset of $\mathbb{R}^2$ by

$$\Gamma = \{(x, y) \in \mathbb{R}^2 \mid x^4 - x^2 + y^2 = 0\}$$

We define open sets in Γ as intersections of Γ with open subsets of $\mathbb{R}^2$. Show that Γ does not admit a structure of a smooth 1-manifold.

1.3. Stereographic projection
Let M be the unit 2-dimensional sphere in $\mathbb{R}^3$, $N, S \in M$, where $N = (0, 0, 1)$ and $S = (0, 0, -1)$. Define $U_N = M \setminus \{N\}$, $U_S = M \setminus \{S\}$, $V_N = V_S = \mathbb{R}^2$. Define also the map $\varphi_N : U_N \to V_N$ in the following way: if $p \in U_N$, the image $\varphi_N(p)$ is the intersection of the line through N and p with the plane $\{z = 0\}$. The map $\varphi_S : U_S \to V_S$ is defined in the same way (substitute N by S everywhere).

Compute explicitely the maps φ_N, φ_S and the transition map $\varphi_N \circ \varphi_S^{-1}$. Show that the collection $(U_\alpha, V_\alpha, \varphi_\alpha)_{\alpha \in \{S,N\}}$ is a smooth atlas, and M is a smooth manifold.

1.4. Introduce a structure of a smooth manifold on

(a) a 2-dimensional torus $\mathbb{T}^2$ obtained from a square $[0, 1] \times [0, 1]$ by identification of the boundary:
$$(0, y) \sim (1, y), \quad (x, 0) \sim (x, 1) \qquad \forall x, y \in [0, 1];$$

(b) a Klein bottle obtained from a square $[0, 1] \times [0, 1]$ by identification of the boundary:

$$(0, y) \sim (1, y), \quad (x, 0) \sim (1 - x, 1) \qquad \forall x, y \in [0, 1];$$

(c) a 3-dimensional torus $\mathbb{T}^3$ obtained from a cube $[0, 1] \times [0, 1] \times [0, 1]$ by identification of the boundary:

$$(0, y, z) \sim (1, y, z), \quad (x, 0, z) \sim (x, 1, z), \quad (x, y, 0) \sim (x, y, 1) \qquad \forall x, y, z \in [0, 1].$$