Riemannian Geometry IV, Homework 10 (Week 10)

10.1. (**Remark 5.19**)
Let (M, g) be a Riemannian manifold, $p \in M$, $v \in T_pM$.

(a) Show that a curve $c(t) = \exp_p(tv)$ is a geodesic.

(b) Show that every geodesic γ through p can be written as $\gamma(t) = \exp_p(tw)$ for appropriate $w \in T_pM$.

10.2. (**Lemma 5.20**)
Let (M, g) be a Riemannian manifold and $p \in M$. Let $\varepsilon > 0$ be small enough such that

$$\exp_p : B_\varepsilon(0_p) \to B_\varepsilon(p) \subset M$$

is a diffeomorphism. Let $\gamma : [0, 1] \to B_\varepsilon(p) \setminus \{p\}$ be any curve.

Show that there exists a curve $v : [0, 1] \to T_pM$, $\|v(s)\| = 1$ for all $s \in [0, 1]$, and a non-negative function $r : [0, 1] \to \mathbb{R}_{\geq 0}$, such that

$$\gamma(s) = \exp_p(r(s)v(s)).$$

10.3. (**Lemma 5.14**) Use the exponential map to show that any vector field $X \in \mathfrak{X}_c(M)$ along a smooth curve $c(t) : [a, b] \to M$ is a variational vector field of some variation $F(s, t)$ (i.e., $X(t) = \frac{\partial F}{\partial s}(0, t)$). Show that if $X(a) = X(b) = 0$ then the variation $F(s, t)$ can be chosen to be proper.

10.4. Geodesic normal coordinates
Let (M, g) be a Riemannian manifold and $p \in M$. Let $\varepsilon > 0$ such that

$$\exp_p : B_\varepsilon(0_p) \to B_\varepsilon(p) \subset M$$

is a diffeomorphism. Let $v_1, \dots, v_n$ be an orthonormal basis of T_pM. Consider a local coordinate chart of M given by $\varphi = (x_1, \dots, x_n) : B_\varepsilon(p) \to V = \{w \in \mathbb{R}^n \mid \|w\| < \varepsilon\}$ via

$$\varphi^{-1}(x_1, \dots, x_n) = \exp_p(\sum_{i=1}^{n} x_i v_i).$$

The coordinate functions $x_1, \dots, x_n$ of φ are called *geodesic normal coordinates.*

(a) Let g_{ij} be the metric in terms of the above coordinate system φ. Show that at the point p

$$g_{ij}(p) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

(b) Let $w = (w_1, \dots, w_n) \in \mathbb{R}^n$ be an arbitrary vector, and $c(t) = \varphi^{-1}(tw)$. Explain why $c(t)$ is a geodesic and deduce from this fact that

$$\sum_{i,j} w_i w_j \Gamma_{ij}^k(c(t)) = 0$$

for all $1 \leq k \leq n$.

(c) Derive from (b) that all Christoffel symbols Γ_{ij}^k of the chart φ vanish at the point p (by choosing appropriate vectors $w \in \mathbb{R}^n$).