Riemannian Geometry IV, Homework 5 (Week 5)

Due date for starred problems: **Thursday, November 26**.

5.1. ($\star$) Let M be a smooth manifold and let $X, Y, Z \in \mathfrak{X}(M)$ be vector fields on M, and let $a \in \mathbb{R}$. Prove the following identities concerning the Lie bracket:

(a) Linearity $[X + aY, Z] = [X, Z] + a[Y, Z]$.

(b) Anti-symmetry $[Y, X] = -[X, Y]$.

(c) Jacobi identity $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$.

The Hairy Ball Theorem. Let $S^n \subset \mathbb{R}^{n+1}$ denote the unit n-sphere. If n is even, then there is no continuous non-vanishing vector field $X \in \mathfrak{X}(S^n)$.

This theorem tells us for example that it can not be windy everywhere at once on Earth's surface – at any given moment, the horizontal wind speed somewhere must be zero.

Exercise 4.4(b) shows that The Hairy Ball Theorem does not hold in odd dimensions. Moreover, it can be generalized in the following way.

5.2. (a) Find a non-vanishing vector field on S^{2m+1} for arbitrary m.

(b) Construct $2m+1$ vector fields on S^{2m+1} forming a basis of T_pS^{2m+1} at every point $p \in S^{2m+1}$.

5.3. Tangent space of a matrix group as a Lie algebra
Let $G \subset M_n(\mathbb{R})$ be a matrix group and $h \in G$. We consider the tangent space T_hG as a subspace of $M_n(\mathbb{R})$.

(a) Let $g(s) \in G$ be a path in G with $g(0) = I$, and let $g'(0) = A \in T_IG \subset M_n(\mathbb{R})$. Let $\gamma(s) = g^{-1}(s)$. Show that $\gamma'(0) = -A$.

(b) Let $g \in G$ and $A \in T_IG \subset M_n(\mathbb{R})$. Show that $gAg^{-1} \in T_IG$. (The map $\mathrm{Ad}_g : T_IG \to T_IG$ sending $A \in T_IG$ to $gAg^{-1} \in T_IG$ is called an *adjoint representation* of G).

(c) Show that the tangent space T_hG at $h \in G$ can be obtained from T_IG by multiplying all the matrices from T_IG by h from the left: $T_hG = hT_IG$. Show that T_hG can also be obtained from T_IG by multiplying all the matrices from T_IG by h from the right.

(d) Show that for every $A \in T_IG$ there exists a vector field $X \in \mathfrak{X}(G)$ with $X(I) = A$.
Hint: try to find a *left-invariant field*, i.e. a field satisfying $X(gh) = gX(h)$ for $g, h \in G$.

(e) Show that if $A, B \in T_IG$, then $[A, B] = AB - BA$ is also an element of T_IG.

Remark: Exercise 5.3 can be generalized to any Lie group, we will see it in the next term.

5.4. ($\star$) Let $\mathbb{H}^2$ be the upper half-plane model of hyperbolic 2-space. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ and define the map

$$f_A : \mathbb{H}^2 \to \mathbb{H}^2, \ f_A(z) = \frac{az+b}{cz+d}.$$

(a) Show that $f_A \circ f_B = f_{AB}$.

(b) Show that for every $A \in SL_2(\mathbb{R})$ the map f_A is an isometry of $\mathbb{H}^2$.
Hint: show first that

$$\mathrm{Im}(f_A(z)) = \frac{\mathrm{Im}(z)}{|cz+d|^2}.$$