Riemannian Geometry IV, Homework 7 (Week 7)

Due date for starred problems: **Thursday, December 10**.

7.1. Covariant derivative in $\mathbb{R}^n$

We define covariant derivative $\nabla_v X$ of a vector field X in the direction of vector $v \in T_p\mathbb{R}^n = \mathbb{R}^n$ at point p in $\mathbb{R}^n$ as

$$(\nabla_v X)(p) = \lim_{t\to 0} \frac{X(p+tv) - X(p)}{t}$$

Show the following properties of the covariant derivative in $\mathbb{R}^n$:

(a) $\nabla_v(X+Y) = \nabla_v(X) + \nabla_v(Y)$;

(b) $\nabla_v(fX) = v(f)X(p) + f(p)\nabla_v X$, where $f \in \mathbb{C}^\infty(\mathbb{R}^n)$, and $v(f)$ denotes the derivative of f in direction v;

(c) $\nabla_{\alpha v+\beta w} X = \alpha\nabla_v X + \beta\nabla_w X$ for $\alpha, \beta \in \mathbb{R}$;

(d) $v(\langle X, Y\rangle) = \langle \nabla_v X, Y\rangle + \langle X, \nabla_v Y\rangle$, where $\langle\cdot,\cdot\rangle$ denotes the Euclidean dot-product, and $\langle X, Y\rangle$ is considered as a smooth function on $\mathbb{R}^n$;

(e) $\nabla_X Y - \nabla_y X = [X, Y]$, where $X, Y, \nabla_X Y, \nabla_Y X \in \mathfrak{X}(\mathbb{R}^n)$, and $(\nabla_X Y)(p)$ is defined as $(\nabla_{X(p)} Y)(p)$.

7.2. ($\star$) Let $\mathbb{H}^n$ be the upper half-space model of hyperbolic n-space,

$$\mathbb{H}^n = \{x \in \mathbb{R}^n \mid x_n > 0\}, \quad g(v, w) = \frac{\langle v, w\rangle}{x_n^2},$$

where $v, w \in T_x\mathbb{H}^n$, and we write g for the metric on $\mathbb{H}^n$ identifying each tangent space canonically with $\mathbb{R}^n$.

Calculate all Christoffel symbols Γ_{ij}^k for the global coordinate chart given by the identity map $\varphi : \mathbb{H}^n \to \mathbb{R}^n$, $\varphi(x) = x$.

7.3. (a) Calculate all Christoffel symbols Γ_{ij}^k for the unit ball model $\mathbb{B}^2$ of hyperbolic plane, again for the global coordinate chart given by the identity map $\varphi : \mathbb{B}^2 \to \mathbb{R}^2$, $\varphi(x) = x$. Recall the the metric is given by

$$g(v, w) = \frac{4}{(1 - \|x\|^2)^2}\langle v, w\rangle$$

(b) Do the same for the unit ball model $\mathbb{B}^n$ of hyperbolic n-space.