

**Riemannian Geometry IV, Homework 8 (Week 8)****Due date** for starred problems: **Thursday, December 10.**

**8.1.** Let  $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$  be the unit sphere inside 3-space, with the induced metric from the standard Euclidean metric on  $\mathbb{R}^3$ .

(a) (★) Let  $c$  be the curve on  $S^2$  given by

$$c(t) = \left( \frac{1}{\sqrt{2}}, \frac{\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}} \right),$$

and let  $v \in T_{c(0)}S^2$  be given by

$$v = (0, 1, 0) \in T_{c(0)}S^2 \subset T_{c(0)}\mathbb{R}^3.$$

Find the unique  $X \in \mathfrak{X}_c(S^2)$  that is parallel along  $c$  and  $X(0) = v$ .

(b) Let  $\gamma_1, \gamma_2 : [0, \pi] \rightarrow S^2$  be two curves connecting the north and south poles  $N$  and  $S$  defined by

$$\gamma_1(t) = (0, \sin t, \cos t)$$

$$\gamma_2(t) = (\sin t, 0, \cos t)$$

Show that the isomorphisms of  $T_N(S^2)$  and  $T_S(S^2)$  given by parallel transports along  $\gamma_1$  and  $\gamma_2$  are different, i.e. find  $u \in T_N(S^2)$  such that  $P_{\gamma_1}(u) \neq P_{\gamma_2}(u)$ .

**8.2.** Let  $\mathbb{H}^2 = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$  be the upper-half plane with its usual hyperbolic metric. Let  $c$  be the curve in  $\mathbb{H}^2$  given by  $c(t) = i + t$  for  $t \in \mathbb{R}$ . Identifying the tangent space to each point of  $\mathbb{H}^2$  in the usual way with  $\mathbb{C}$ , find the parallel vector field  $X(t) \in \mathbb{C} = T_{c(t)}\mathbb{H}^2$  along  $c$ , which is determined by its value at  $t = 0$ :

$$X(0) = 1 \in \mathbb{C} = T_i\mathbb{H}^2.$$

**8.3.** Let  $(M, g)$  be a Riemannian manifold and  $c : [a, b] \rightarrow M$  be a smooth curve. Let  $\frac{D}{dt}$  denote the corresponding covariant derivative along the curve  $c$ . Let  $X, Y$  be any two parallel vector fields  $X, Y$  along  $c$ . Show that

$$\frac{d}{dt} \langle X, Y \rangle \equiv 0,$$

i.e., the parallel transport  $P_c : T_{c(a)}M \rightarrow T_{c(b)}M$  is a linear isometry.

(a) (★) Prove this statement in the particular case when the vector fields  $X, Y$  along  $c$  have global extensions  $\tilde{X}, \tilde{Y} : M \rightarrow TM$ .

(b) Do the same computation for a general case writing  $X(t), Y(t)$  in local coordinates.

**8.4.** Given a curve  $c : [a, b] \rightarrow \mathbb{R}^3$ ,  $c(t) = (f(t), 0, g(t))$  without self-intersections and with  $f(t) > 0$  for all  $t \in [a, b]$ , let  $M \subset \mathbb{R}^3$  denote the surface of revolution obtained by rotating this curve around the  $z$ -axis. Let  $\nabla$  denote the Levi-Civita connection of  $M$ . An almost global coordinate chart is given by  $\varphi : U \rightarrow V := (a, b) \times (0, 2\pi)$ ,

$$\varphi^{-1}(x_1, x_2) = (f(x_1) \cos x_2, f(x_1) \sin x_2, g(x_1)).$$

(a) Calculate the Christoffel symbols of this coordinate chart and express  $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}$  in terms of the basis  $\frac{\partial}{\partial x_k}$ .

(b) Let  $\gamma_1(t) = \varphi^{-1}(x_1 + t, x_2)$ . Calculate

$$\frac{D}{dt} \gamma'_1,$$

where  $\frac{D}{dt}$  denotes the covariant derivative along  $\gamma_1$ . Show that this vector field along  $\gamma_1$  vanishes if and only if the generating curve  $c$  of  $M$  is parametrized proportional to arc-length. Note that  $\gamma_1$  is obtained by rotation of  $c$  by a fixed angle. Derive from these facts that meridians of a surface of revolution are geodesics if they are parametrized proportional to arc length.

(c) Let  $\gamma_2(t) = \varphi^{-1}(x_1, x_2 + t)$ . Calculate

$$\frac{D}{dt} \gamma'_2,$$

where  $\frac{D}{dt}$  denotes the covariant derivative along  $\gamma_2$ . Show that this vector field along  $\gamma_2$  vanishes if and only if  $f'(x_1) = 0$ . Explain that this implies that parallels of a surface of revolution are geodesics if they have locally maximal or minimal radius.