Riemannian Geometry IV, Homework 9 (Week 9)

9.1. First Variation Formula of energy.
Let $F : (-\varepsilon, \varepsilon) \times [a, b] \to M$ be a variation of a smooth curve $c : [a, b] \to M$ with $c'(t) \neq 0$ for all $t \in [a, b]$, and let X be its variational vector field. Let $E : (-\varepsilon, \varepsilon) \to \mathbb{R}_+$ denote the associated energy, i.e.,

$$E(s) = \frac{1}{2} \int_a^b \| \frac{\partial F}{\partial t}(s, t) \|^2 dt.$$

(a) Show that

$$E'(0) = \langle X(b), c'(b) \rangle - \langle X(a), c'(a) \rangle - \int_a^b \langle X(t), \frac{D}{dt} c'(t) \rangle dt.$$

Simplify the formula for the cases when

(b) c is a geodesic,

(c) F is a proper variation,

(d) c is a geodesic and F is a proper variation.

Let $c : [a, b] \to M$ be a curve connecting p and q (not necessarily parametrized proportional to arc length). Show that

(e) $E'(0) = 0$ for every proper variation implies that c is a geodesic.

(f) Assume that c minimizes the energy amongst all curves $\gamma : [a, b] \to M$ which connect p and q. Then c is a geodesic.

9.2. Rescaling Lemma.
Let $c : [0, a] \to M$ be a geodesic, and $k > 0$. Define a curve γ by

$$\gamma : [0, a/k] \to M, \qquad \gamma(t) = c(kt)$$

Show that γ is geodesic with $\gamma'(t) = kc'(kt)$.

9.3. Let M be a smooth manifold, let $\mathfrak{X}(M)$ be the vector space of smooth vector fields on M, and ∇ be a *general* affine connection (we do not require a Riemannian metric on M and the "Riemannian property", neither the "torsion-free property" of the Levi-Civita connection). We say a map

$$A : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to C^\infty(M) \text{ or } \mathfrak{X}(M)$$

is a *tensor* if it is linear in each argument, i.e.,

$$A(X_1, \cdots, fX_i + gY_i, \cdots, X_r) = fA(X_1, \cdots, X_i, \cdots, X_r) + gA(X_1, \cdots, Y_i, \cdots, X_r),$$

for all $X, Y \in \mathfrak{X}(M)$ and $f, g \in C^\infty(M)$.

(a) Show that

$$T : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M), \qquad T(X, Y) = [X, Y] - (\nabla_X Y - \nabla_Y X)$$

is a tensor (called the *torsion* of the manifold M).

(b) Let

$$A : \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{r \text{ factors}} \to C^\infty(M)$$

be a tensor. The covariant derivative of A is a map

$$\nabla A : \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{r+1 \text{ factors}} \to C^\infty(M),$$

defined by

$$\nabla A(X_1, \ldots, X_r, Y) = Y(A(X_1, \ldots, X_r)) - \sum_{j=1}^{r} A(X_1, \ldots, \nabla_Y X_j, \ldots, X_r).$$

Show that ∇A is a tensor.

(c) Let (M, g) be a Riemannian manifold and $G : \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$ be the Riemannian tensor, i.e., $G(X, Y) = \langle X, Y \rangle$. Calculate ∇G. What does it mean that $\nabla G \equiv 0$?