Riemannian Geometry IV
Christmas Homework

Snowed problems to be sent to Santa by Thursday, December 24th

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Definition. Let (M, g) be a Riemannian manifold.
Let $\varphi : U \to V$ be a chart and $A \subseteq U \subseteq M$ be a subset.
A volume of A is defined by

$$Vol(A) = \int_A d\ Vol = \int_{\varphi(A)} \sqrt{det(g_{ij}))} dx,$$

where g_{ij} is the metric g written in the chart φ.

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Example. Find the volume of the rectangle

$$A = \{z = x + iy \in \mathbb{C} \mid c \le x \le d, a \le y \le b\}$$

in the upper half-plane model of hyperbolic plane $(\mathbf{H}^2, g)$.

Solution. Recall, $\mathbf{H}^2 = \{z \in \mathbb{C} \mid y > 0\}$, and $\langle v, w \rangle_z = \frac{v_1 w_1 + v_2 w_2}{y^2}$.
(we are working in one coordinate chart $\varphi(x, y) = (x, y)$).
Since $(g_{ij}) = \begin{pmatrix} 1/y^2 & 0 \\ 0 & 1/y^2 \end{pmatrix}$, we have $\sqrt{det(g_{ij})} = \frac{1}{y^2}$.
Hence, $Vol(A) = \int\limits_c^d \int\limits_a^b \frac{1}{y^2} dy dx = \int\limits_c^d (\frac{1}{a} - \frac{1}{b}) dx = (d - c)(\frac{1}{a} - \frac{1}{b})$.

Remark:
Note that not all the sides of A are geodesic in hyperbolic metric!

Christmas problems:

1. (∗) Christmas tree.

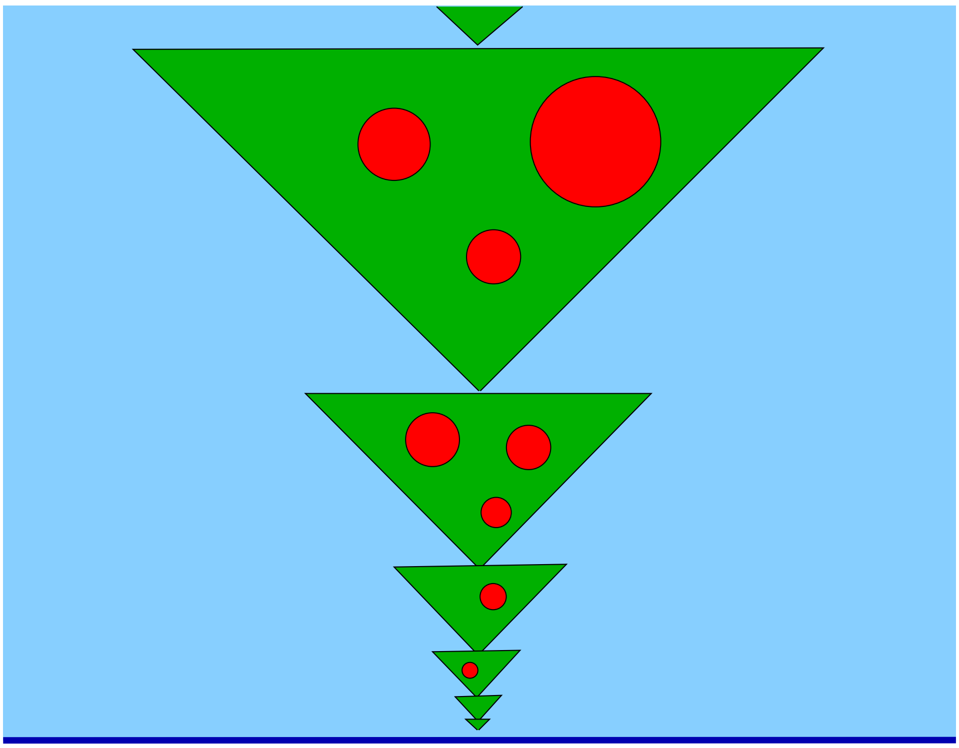
A <u>Christmas tree</u> in $\mathbf{H}^2$ is the union of the domains Δ_k, $k \in \mathbf{Z}$, where Δ_k is a Euclidean right-angled triangle with vertices

$$2^k i,\ 2^k(2i-1),\ 2^k(2i+1).$$

(Note, it is not a hyperbolic triangle, its sides are not hyperbolic geodesics!)

(a) Prove that $Vol(\Delta_k)$ does not depend on k.

(b) Find $Vol(\Delta_k)$.



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2. (∗) Christmas fireworks.

Hyperbolic Santa organises Christmas fireworks in the upper half-plane. He decided to use exponential map: he chooses a point $p \in \mathbb{H}^2$, an integer number n and a bunch of unit vectors $\{v_k\}$ in $T_p\mathbb{H}^2$ so that $v_k = e^{k\pi i/n}$, then he finds their images $exp_p(v_i)$ and puts lights at the obtained points. Sketch the design Santa got.

3. (∗) Decorating the tree.

Santa (the hyperbolic one) decorates the tree (see question 1) with hyperbolic balls. He wants to arrange the decoration in such a way that the following two conditions were satisfied:

i. Each Δ_k contains at least one ball;

ii. Volume of the union of all balls is finite.

Can you suggest Santa an example of such a collection of balls?

Remark: A hyperbolic ball of radius r centered at $z \in \mathbf{H}^2$ is the set of points in hyperbolic distance at most r from the point z.

Hints:

1. Show that balls centered in the center of the Poincaré disk model $\mathbf{B}^2$ are Euclidean balls.
2. Use the isometry $-i\frac{z+1}{z-1} : \mathbf{B}^2 \to \mathbf{H}^2$ to see that balls centered at $i \in \mathbf{H}^2$ are Euclidean balls. Note that the hyperbolic center of a ball does not coincide with the Euclidean one.
3. Use the isometries $z \to z + a$ and $z \to \alpha z$ ($a, \alpha \in \mathbb{R}$, $\alpha > 0$) to see that all balls in $\mathbf{H}^2$ are Euclidean balls.
4. Let B be a ball centered at the center of the disk model $\mathbf{B}^2$. Find the volume of B as a function of its Euclidean radius r_E. What is its hyperbolic radius r? Find the volume of B as a function of its hyperbolic radius r.
5. Take a family of balls $B_k \in \mathbf{B}^2$ of volume at most $\frac{1}{2^k}$ and use isometries to map these balls into $\Delta_{\pm k}$.
6. What is the largest ball which fits into Δ_k?

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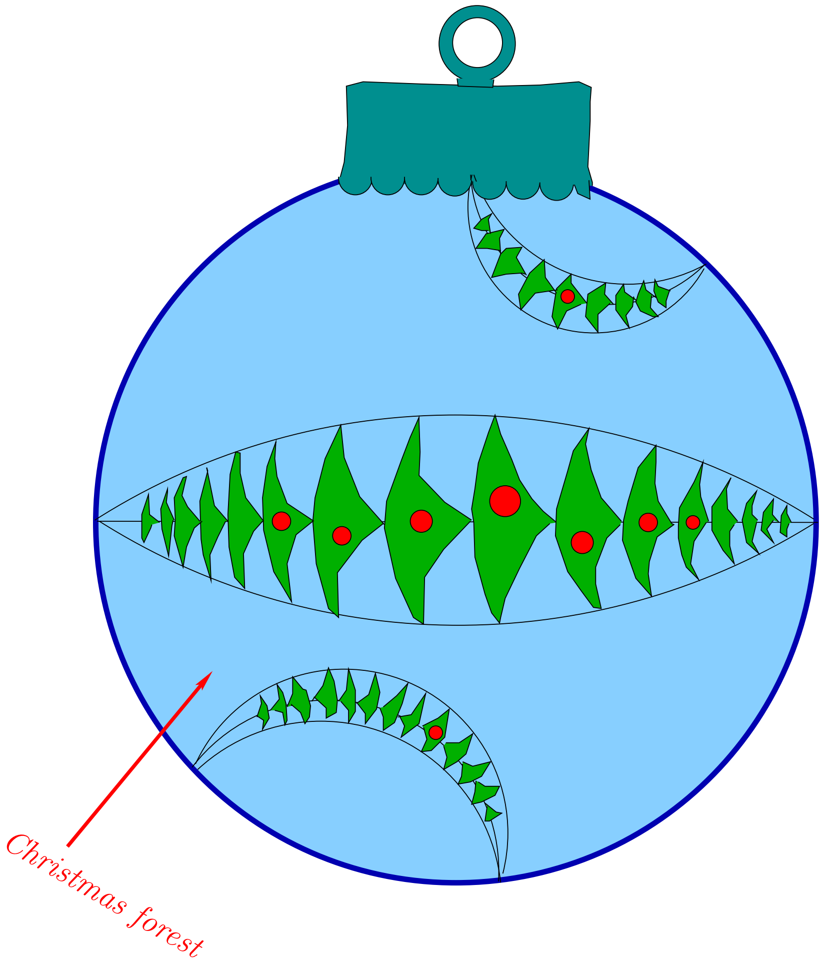
4. (∗) Santa in Durham.

Hyperbolic Santa comes to Durham. Find the places where he will feel most at home (i.e. where the place looks "most hyperbolic")

(a) at the Department of Mathematical Sciences;

(b) at the Durham Cathedral.



Merry Christmas and Happy New Year!