

# Farey's legacy in frieze patterns and discrete geometry

ICMS, 20 April to 24 April 2026

## Schedule

	Monday 20th	Tuesday 21st	Wednesday 22nd	Thursday 23rd	Friday 24th
09.00-09.30	Welcome				
09.30-10.20	Sergey Fomin	Richard Schwartz	Boris Springborn	Alexander Veselov	Karin Baur
10.20-11.00	Coffee break				
11.00-11.50	Eleonor Faber	Katherine Stange	Esther Banaian	Muhammed Uludag	Deepanshu Prasad Anja Sneiderger
12.00-12.50	Emily Gunawan	Ivan Rasskin	Antoine de St. Germain	Thomas Garrity	Sergei Tabachnikov
12.50-14.30	Lunch	Lunch and posters		Lunch	Lunch
14.30-15.20	Michael Cuntz	Matthew Litman Alexander Thomas		Emine Yildirim	
15.20-16.00	Coffee Break	Coffee Break		Coffee Break	
16.00-16.25	Katie Waddle	Rebecca Sheppard		Yefei Ma	
16.30-16.55	Francesca Fedele	Davide Dal Martello		Andrei Zabolotskii	
17.00-18.00	Reception and artwork display				
18.00-21.00				Dinner	

## Titles and Abstracts

**Esther Banaian** (U of California - Riverside)

Orderings of  $k$ -Markov numbers

The  $k$ -Markov numbers, introduced by Gyoda and Matsushita, are positive integers which appear in solutions to the equation  $x^2 + y^2 + z^2 + k(xy + xz + yz) = (3 + 3k)xyz$ . When  $k$  is 0, this recovers the famous Markov equation, a topic of frequent study for more than a century. An impetus for the study of Markov numbers is Frobenius' unicity conjecture. One approach to proving the unicity conjecture is demonstrating that the Markov numbers provide a total ordering on rational numbers in the interval  $[0, 1]$ . With this perspective, Lee-Li-Rabideau-Schiffler and Huang made progress towards the unicity conjecture using techniques from the theory of cluster algebras. By working with generalized cluster algebras, in this talk we consider the orderings on rational numbers induced by  $k$ -Markov numbers as  $k$  varies. We see some similarities and surprising differences. This is based on arXiv:2512.04026 and ongoing joint work with Min Huang.

**Karin Baur** (Ruhr U Bochum & U of Leeds)

Frieze patterns via Plücker coordinates

In this talk, I will show how to obtain  $SL_k$ -frieze patterns using Plücker coordinates by taking certain subcategories of the Grassmannian cluster categories. These are cluster structures associated to the Grassmannians of  $k$ -dimensional subspaces in  $n$ -space. While many of these patterns arise from specialising a cluster to 1, there are also other types of  $SL_k$ -friezes. We will show some examples. In general, it is an open question how they arise.

**Micheal Cuntz** (U of Hannover)

Frieze patterns and simplicial arrangements

Arrangements of lines which triangulate the plane, the so-called simplicial arrangements, appear to be rare. They have been collected during the last 80 years. It is conjectured that they are all known. In my talk I will explain a relation between simplicial arrangements and frieze patterns.

**Davide Dal Martello** (U of Padua)

A novel take on topography

Conway's topograph is an amusing – and powerful – graphical tool devised to capture both the values of a given quadratic form and its whole equivalence class. We revisit this concept via modern combinatorial lenses, developing a much more general framework that leads to striking applications beyond the original representation problem for quadratic forms (e.g., reduction theory and Painlevé equations)

**Antoine de Saint Germain** (The U of Hong Kong)

Positive integer points on Mordell-Schinzal surfaces

The set of positive integer points of the celebrated Markov surface admits the structure of a 3-regular tree. Motivated by this, I will show that the set of positive integer points of each Mordell-Schinzal surface admits the structure of a 2-regular graph. Whether this graph has finitely many vertices is determined by whether or not an associated algebraic system (a generalisation of a frieze pattern) is periodic or not. Using this observation, I will describe a resolution of the positive Mordell-Schinzal problem. This is partly based on ongoing joint work with Robin Zhang (MIT).

**Eleonor Faber** (Graz/Leeds)

Additive frieze patterns and polytopes

Frieze patterns of integers were first studied by Coxeter and Conway-Coxeter in the 1970s from a combinatorial perspective. Since the 2000s they had a remarkable renaissance in cluster theory, in particular, in connection with cluster categories of type A.

In this talk, we consider the variant of nonnegative additive frieze patterns. Our goal is to find a representation theoretic interpretation of these friezes through a root category as well as a way to enumerate them. Therefore we consider associahedra coming from the cluster category, which lead us to a construction of an associahedron for our root category.

This is joint work with Asilata Bapat, Véronique Bazier-Matte, Kunda Kambaso, Bethany Marsh, and Yadira Valdivieso.

**Francesca Fedele** (Leeds)

Infinite superfriezes

Conway–Coxeter friezes associated with triangulated polygons extend to periodic infinite friezes on the annulus, as shown by Baur, Parsons, and Tschabold. Morier-Genoud, Ovsienko, and Tabachnikov introduced a supersymmetric analogue of classic friezes, and these were later related to Penner–Zeitlin’s decorated super-Teichmüller theory by Musiker, Ovenhouse, and Zhang: in this setting, they arise from oriented triangulations of the disc. In this joint project with A. Burcroff, İ. Çanakçı, M. Garcia and V. Klász, we extend superfriezes to the infinite setting and relate them to triangulations of annuli.

**Sergey Fomin** (U of Michigan)

Heronian friezes

A central problem of distance geometry asks for reconstruction of all measurements associated with a point configuration from a small subset of measurements. I will discuss the connections between this problem, the combinatorics of triangulated polygons, and the theory of friezes. This is joint work with Linus Setiabrata.

**Thomas Garrity** (Williams College)

A ternary Farey tree, integer partitions and double Farey boats

**Emily Gunawan** (U of Massachusetts Lowell)

Heap Birkhoff polytopes

For each orientation  $c$  of a type  $A$  Dynkin quiver, we define a  $c$ -Birkhoff polytope  $\text{Birk}(H)$  and show that it is integrally equivalent to the order polytope for poset  $H$ , the heap of the  $c$ -sorting word of the longest element  $w_0$ . A consequence of this result is that the volume of the  $c$ -Birkhoff polytope is the number of the longest chains in the type  $A$   $c$ -Cambrian lattice. We will also discuss current work in type  $B$  and a generalization of our result to other Birkhoff subpolytopes  $\text{Birk}(H)$  corresponding to heaps  $H$  of other reduced words of an element in the symmetric group.

**Matthew Litman** (University College Dublin)  
Loxodromic Sequences of Tangent Circles in Polytopal Packings

In the 1960s, Coxeter initiated a study into infinite sequences of circles in the plane having the property that every 4 consecutive elements of a sequence are mutually tangent, that is they form a Descartes quadruple. He went on to show that if their bends fall into geometric progression, then there is a unique such sequence and the ratio between consecutive bends is equal to  $\varphi + \sqrt{\varphi}$ , where  $\varphi$  is the golden ratio. This spiral sequence of circles is mapped onto itself by a dilative-rotation whose dilation factor is  $\varphi + \sqrt{\varphi}$  and angle of rotation is related to the degree 4 equation satisfied by  $\varphi + \sqrt{\varphi}$ . Some 50 years later, the theory of polytopal circle packings was developed, whereas Descartes configurations correspond to the tetrahedron. In this talk, we will extend Coxeter's notion of loxodromic sequences of tangent circles to the polytopal setting, show that for each regular polytope there is a unique sequence where the bends fall into geometric progression, and provide explanations to some of Coxeter's observations which are obscured by the elegant algebraic properties of the golden ratio.

**Yefei Ma** (U of Montpellier)  
Snake graphs in higher dimensions

In this poster I would like to discuss several possible generalisations of snake graphs to the multidimensional case. The aim is to compare their properties and their relation to Cohn matrices, generalised Markov numbers and geometry of numbers.

**Deepanshu Prasad** (International Center for Mathematical Sciences - Sofia)  
Growth of infinite T-friezes of affine type

We investigate the growth coefficients of infinite  $T$ -friezes arising in the context of cluster-tilted algebras of tame type. We prove that the growth coefficients for those friezes given by the non-homogeneous stable tubes all have the same coefficient. We use the Caldero-Chapoton (CC) map, a technique widely employed in the literature, to construct infinite frieze patterns. We also provide an explicit formula for the  $k$ -th growth coefficient, expressed directly in terms of data from homogenous tubes. Finally, we give examples to illustrate our main results. This is joint work with K. Baur, A. Felikson, P. Tumarkin, and E. Yıldırım.

**Ivan Rasskin** (Aix-Marseille U)  
Rational links in Farey complexes

Rational tangles were introduced by Conway as a family of tangles in bijection with the rational numbers. This correspondence can be described algebraically by relating the construction of rational tangles to the operations involved in continued fraction expansions. In this talk, we will explore various ways to visualize this connection through some generalizations of the Farey triangulation. These perspectives provide tools for deriving upper bounds on geometric invariants of rational links.//

**Richard Schwartz** (Brown U)  
Farey addition, Pappus's theorem, and bendings of Farey tiling

If you don't learn how to correctly add fractions in grade school, you might end up using Farey addition to add them. This "adding mistake" turns out to be an important operation in its own right, closely connected to continued fractions and the modular group. In this talk I will explain how to view Pappus's Theorem, a classic theorem in projective geometry, as a deformation of Farey addition. I will use this viewpoint to construct patterns of geodesics in the higher rank symmetric space  $SL_3(\mathbb{R})/SO(3)$  which have the same combinatorial structure as the Farey tiling of the hyperbolic plane, except that they are geometrically bent. One can view these as something like higher rank pleated surfaces.

**Rebecca Sheppard** (U of Manchester)

Discrete circles and their intersections in integer geometry

Integer geometry explores objects whose vertices lie on the integer lattice with congruence defined by lattice-preserving affine transformations. These objects are a possible discrete analogue for circles in Euclidean geometry, invariant in integer geometry, which challenge our geometric intuition. Unlike their classical counterparts, circles in integer geometry are unbounded, exhibit nontrivial arithmetic structure, and possess positive density in the plane. I will discuss pre-requisite notions in integer geometry for those unfamiliar with the area before introducing newer concepts of circles. The densities of integer circles and their intersections have expressions in the form of Euler products.

**Anja Sneperger** (U of Leeds)

Generalization of Heronian friezes

Motivated by computational geometry of point configurations on the Euclidean plane, and by the theory of cluster algebras of type  $A$ , Sergey Fomin and Linus Setiabrata have recently introduced Heronian friezes - the Euclidean analogues of Coxeter's frieze patterns. I will use Plucker relations in the Grassmannian  $\text{Gr}(3,n)$  to give relations that hold amongst some of the entries of the Heronian frieze of order  $n$ . Furthermore, I will present a connection between certain subfriezes of a Heronian frieze and Plucker friezes  $P(3,n)$ , and then show that some determinants of the matrices whose elements lie in those subfriezes are vanishing.

**Boris Springborn** (TU Berlin)

Diophantine approximation and the hyperbolic geometry of numbers beyond the Farey triangulation and continued fractions

For a geometric understanding of the worst approximable rational and irrational numbers, it helps to consider not only the Farey triangulation of the hyperbolic plane, but all ideal triangulations of the modular torus.

**Katherine Stange** (U of Colorado Boulder)

Starscapes and Schmidt Arrangements

I will discuss two generalizations of the Farey tessellation (sort of like two shorter talks). The first is a generalization to algebraic integers of fixed degree. Here, generalization is used loosely, because we mainly study the structure of these sets of integers in a way that suggests generalization, without actually providing such generalizations. The second is a generalization to imaginary quadratic fields that is originally due to Asmus Schmidt, and which has connections to Apollonian circle packings. Both of these are very visual explorations, so the talk will have plenty of pretty pictures, hyperbolic geometry, and open questions.

**Sergei Tabachnikov** (Pennsylvania State U)

Continuous 2-friezes patterns

I shall define a continuous version of 2-frieze patterns, a combinatorial structure related with Coxeter and Conway frieze patterns and  $SL(k)$ -tilings. I shall describe the relation of continuous 2-friezes with the moduli space of projective curves and relate the (pre)symplectic structure on the space of closed 2-friezes, considered as a cluster variety, with the Adler-Gelfand-Dikii bracket on the space of 3rd order differential operators.

**Alexander Thomas** (U of Lyon 1)

Deformed Farey triangulation and Springborn operations

The modular group acts by symmetries on the classical Farey triangulation. The embedding of the modular group into the isometry group of the hyperbolic plane allows a 1-parameter family of deformations. Using another embedding gives rise to a deformed Farey tessellation, which is intimately linked to  $q$ -deformed rational numbers, introduced by Morier-Genoud and Ovsienko. The geometric picture suggests to consider the set of  $q$ -rationals as a 1-dimensional circle packing. A new operation, a quadratic version of the Farey sum which we call the Springborn sum, naturally arises in this viewpoint. The construction of a 2-dimensional extension of the circle packing is work in progress. Joint with Perrine Jouteur and Olga Paris-Romaskevich.

**Muhammed Uludag** (Galatasaray U)

Recursively defined functions on the Farey tree and the codenominator

Codenominator is a function defined recursively on the Farey tree and yields an extension of the Fibonacci sequence to rationals as an integral-valued map. After discussing its arithmetic properties, we will show how it is related to Dyer's outer automorphism of the group  $PGL(2, \mathbb{Z})$  and the involution Jimm of the real line. Automorphisms of the cubic tree studied by Djokovic–Miller and Conder provide analogues of the codenominator. We will also explain its connection to quantizations of real numbers recently introduced by Ovsienko and Mourier-Genoud. The talk will provide many questions and conjectures related to these functions.

**Alexander Veselov** (Loughborough U)

Springborn's variation on the Farey theme and exceptional slopes

In 2024 Boris Springborn introduced the notion of the Markov fractions by ingeniously modifying the Farey mediant. I will show that the Markov fractions are precisely the slopes of the exceptional vector bundles on  $\mathbb{P}^2$  studied in 1980s by Drèzet and Le Potier and by Rudakov. In particular, this provides a simpler proof of Rudakov's result claiming that the ranks of the exceptional bundles on  $\mathbb{P}^2$  are Markov numbers.

**Katie Waddle** (U of Michigan)

Geometric interpretations of Coxeter–Conway friezes

Three-term relations of the form  $AB + CD = EF$  arise in multiple mathematical contexts, including the Ptolemy equation for a cyclic quadrilateral, Casey's theorem on bitangents, Penner's relation for lambda lengths, and Plücker's identity for the maximal minors of a  $2 \times 4$  matrix. This talk will provide historical context for these relations, explain how these different occurrences can be directly obtained from each other, and discuss a connection to cross-ratios.

**Emine Yıldırım** (International Center for Mathematical Sciences - Sofia)

A mirror deformation of Markov numbers

We introduce a natural and new  $q$ -deformation of the Markov numbers, called mirror Markov numbers, arising from a deformed squared Markov equation whose solutions specialize to the squares of the classical Markov numbers. We characterize these numbers and define a mutation rule, mirror mutation, that generates them all, together with a geometric realization on a once-punctured sphere with three orbifold points. Our construction yields deformations of the Fibonacci and Pell branches with explicit formulas and recovers several well-known generalized Markov equations by specialization. In particular, we obtain the super Markov numbers and use them to prove a conjecture of Musiker. This is a joint work with L. Bittmann, P. Jouteur, E. Kantarci Oguz, and M. Molander.

**Andrei Zabolotskii** (The Open U)

Maximally non-tame  $SL_2$ -tilings

Like frieze patterns,  $SL_2$ -tilings satisfy the diamond rule. Like frieze patterns,  $SL_2$ -tilings come from paths in the Farey graph, but only the tame ones. Much less is known about wild (non-tame)  $SL_2$ -tilings. We will demonstrate  $SL_2$ -tilings that are maximally non-tame: we prove that the maximum wild density of an integer  $SL_2$ -tiling is  $2/5$  and present  $SL_2$ -tilings over  $\mathbb{Z}/N\mathbb{Z}$  with wild density 1. Moreover, the wildest integral  $SL_2$ -tilings is also maximally non-tame in terms of rank.