Surfaces with conical singularities: Fukaya categories as A_∞-localizations

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The topological Fukaya category of a smooth surface

In [HKK17], an A_∞-category was associated to any graded surface S with stops. It was shown by Bocklandt that it computes the partially wrapped Fukaya category of S, as defined by Abouzaid–Seidel, and it has the advantage to enjoy a simpler description.

A_∞-categories

An A_∞-category over a base ring K is the data of a class of objects, and of morphism spaces endowed with a structure of K-complexes. The associativity of the composition is not required but instead a family of higher multiplications μ^n is given, and they must satisfy the Stasheff relations:

$$\sum_{\substack{0\leq k,\ 1\leq j \\ k+j\leq n}} (-1)^{\dagger}\mu^{n-j+1}(a_n,\dots,a_{k+j+1},\mu^j(a_{j+k},\dots,a_{k+1}),a_k,\dots,a_1)=0.$$

In analogy with the construction of the homotopy category of an abelian category, one can construct the category of twisted complexes $Tw\mathcal{A}$ over $\mathcal{A}$, whose homotopy category $\mathcal{A}^{tr} := H^0 Tw\mathcal{A}$ is triangulated.

Minimal A_∞-category of an arc system

A graded marked surface (S, M, η) is the data of a smooth surface S with boundary and a set of stops M on its boundary, endowed with a line field η. The Fukaya category $\mathcal{F}(S)$ is obtained by taking the category of twisted complexes of an A_∞-category $\mathcal{F}_A(S)$, whose objects are given by a collection of compatible graded arcs A on the surface, and whose morphism spaces reflect intersections. See the left hand side of Figure 1 for an example of a marked surface with a collection of compatible arcs. The higher multiplications are defined in such a way that distinct arc collections give Morita equivalent categories. The construction therefore depends solely on the graded marked surface.

Gentle Algebras

Gentle algebras are given by quivers with relations enjoying a simple description. Their representation theory has been widely studied and is very well understood.

They appear in this context as endomorphism rings of formal generators of the Fukaya category. Given a graded (homologically smooth and proper) gentle algebra Λ, one can associate a graded marked surface (S, M, η) with admissible dissection A, such that there is a triangle equivalence:

$$per(\Lambda) \simeq \mathcal{F}_A(S)^{tr},$$

where $per(\Lambda)$ denotes the perfect derived category of Λ [LP20]. This has led to several research works, one of which is the establishment of a complete derived invariant for graded gentle algebras.

Surfaces with conical singularities

By definition, a graded marked surface with conical singularities S, also called a pinched surface, is given by the contraction of a collection C of simple closed curves of winding number zero, on a smooth graded marked surface $\hat{S}$. See the right hand side of Figure 1 for an illustration.

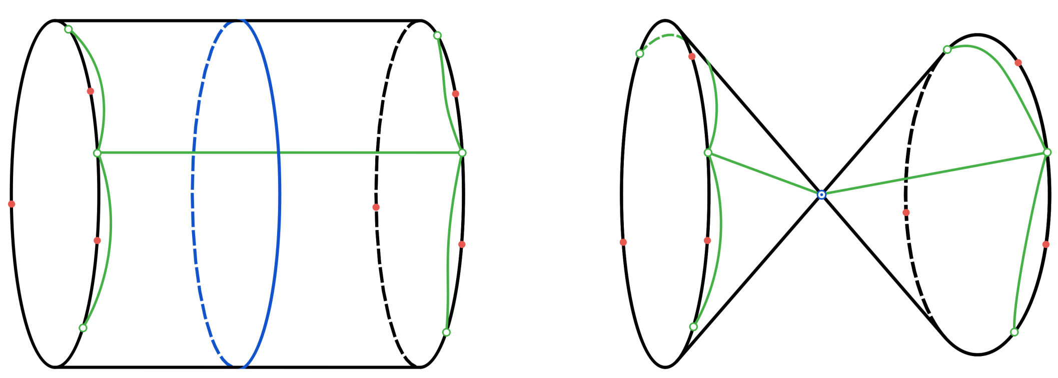


Figure 1. A marked surface with a simple closed curve (blue), and the marked surface with conical singularities obtained by contraction.

One would like to define the topological Fukaya category $\mathcal{F}(S)$ of a singular surface S in analogy with the smooth case. Moreover $\mathcal{F}(S)$ should be related to $\mathcal{F}(\hat{S})$ by some categorical operation. In the case of orbifold singularities, this has been achieved through the use of skew-group A_∞-categories [AP24]. In [J22], following a suggestion of Auroux, the author defined the topological Fukaya category of S by taking a A_∞-quotient of the Fukaya category of a smooth surface. Our work can be seen as an algebraic analogous construction in which we seek an explicit description of the localization.

Let S be graded marked surface with conical singularities. Motivated by the idea that the Fukaya category should be generated by a finite collection of arcs on the surfaces, we define the notion of an admissible dissection A on S.

- A is a collection of pairwise non isotope arcs, without intersections in the interior,
- A cuts S into polygons, each containing exactly one unmarked boundary segment,
- Elements of A intersect only cyclically at the singularities.

Defined in this way, an admissible dissection A on S lifts naturally to an admissible dissection $\hat{A}$ on the smooth surface $\hat{S}$. The Fukaya category of S is then defined to be the A_∞-localization $\mathcal{D}(Tw\mathcal{F}_{\hat{A}}(\hat{S})|\mathcal{B})$, of $Tw\mathcal{F}_{\hat{A}}(\hat{S})$ by the full subcategory $\mathcal{B}$ generated by the spherical objects corresponding to the simple closed curves in C.

Minimal model and formality of the A_∞-localization

One difficulty arising from this definition comes from the intricacy of the morphism spaces and higher multiplications after localization.

A_∞-localization

For a full subcategory $\mathcal{B}$ of $\mathcal{A}$ supported on an object B, the A_∞-category $\mathcal{D} := \mathcal{D}(\mathcal{A}|\mathcal{B})$ as the same objects as $\mathcal{A}$, and morphisms are given by sequences of morphisms of $\mathcal{A}$ passing through B. The higher multiplications are given by:

$$\mu^r_{\mathcal{D}}(a_{n_r}\cdot\dots\cdot a_{n_{r-1}+1},\dots,a_{n_1}\cdot\dots\cdot a_1) = \sum_{n_r\geq k+j\geq n_{r-1}+1} (-1)^{\dagger} a_{n_r}\cdot\dots\cdot a_{k+j+1}\cdot\mu^{j+1}_{\mathcal{A}}(a_{k+j},\dots,a_k)\cdot a_{k-1}\cdot\dots\cdot a_1$$

When K is a field, it enhances Verdier quotient: $\mathcal{D}(\mathcal{A}|\mathcal{B})^{tr} \simeq \mathcal{A}^{tr}/\mathcal{B}^{tr} \simeq per(\Lambda)/thick(B)$.

Minimal model

By a theorem of Kadeishvili, any A_∞-category $\mathcal{A}$ admits a quasi-equivalent A_∞-structure on its homology. The higher multiplications are given inductively by:

$$\mu^n(a_n,\dots,a_1) = \sum_{2\leq r}\ \sum_{\substack{1\leq s_1,\dots,s_r \\ s_1+\dots+s_r=n}} T^1(\mu^r_{\mathcal{A}}(\mathcal{F}^{s_r}(a_n,\dots,a_{n-s_r+1}),\dots,\mathcal{F}^{s_1}(a_{s_1},\dots,a_1))),$$

$$\mathcal{F}^n(a_n,\dots,a_1) = \sum_{2\leq r}\ \sum_{\substack{1\leq s_1,\dots,s_r \\ s_1+\dots+s_r=n}} \mathcal{G}^1(\mu^r_{\mathcal{A}}(\mathcal{F}^{s_r}(a_n,\dots,a_{n-s_r+1}),\dots,\mathcal{F}^{s_1}(a_{s_1},\dots,a_1))),$$

where $\mathcal{F}^1$ is the inclusion of $H^*\mathcal{A}$, $\mathcal{G}^1$ a projection on $H^*\mathcal{A}$, and T^1 a contracting homotopy. Any such homotopy transfer is called a minimal model of $\mathcal{A}$.

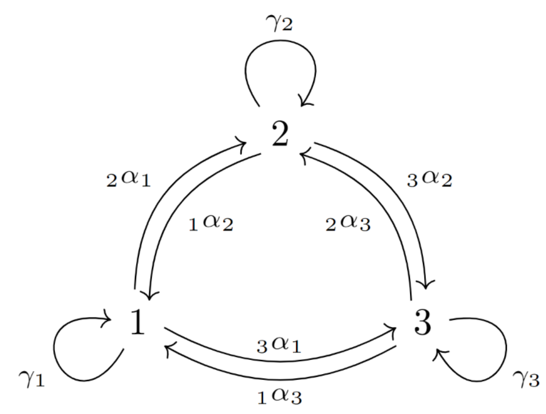
It has been shown that $\mathcal{D}(H^*\mathcal{F}_{\hat{A}}(\hat{S})|B)$ is a formal A_∞-category, meaning that it is quasi-equivalent to its homology, where $H^*\mathcal{F}_{\hat{A}}(\hat{S})$ is endowed with an A_∞-structure obtained by homotopy transfer [B24].

The topological Fukaya category of a pinched surface

The previous result motivates the definition $\mathcal{F}_A(S) = \mathcal{D}(H^*\mathcal{F}_{\hat{A}}(\hat{S})|B)$. The topological Fukaya category $\mathcal{F}(S)$ of the graded marked surface S with conical singularities is then obtained by taking the category of twisted complexes over $\mathcal{F}_A(S)$. By construction, it enjoys the following property:

$$\mathcal{F}(S) \simeq \mathcal{D}(\mathcal{F}(\hat{S})|\mathcal{B}).$$

Moreover, $\mathcal{F}_A(S)$ can by easily described by quiver with relations. Here's a typical example around arcs going through a singularity:



$(a)\ \gamma_{k\,k}\alpha_i = {}_k\alpha_i\gamma_i$

$(b)\ {}_i\alpha_{k\,k}\alpha_i = \gamma_i^2 - e_i$

$(c)\ {}_k\alpha_{(k-1)}\cdots{}_{(i+1)}\alpha_i(\gamma_i - e_i)^{l^+}$
$\quad = {}_k\alpha_{(k+1)}\cdots{}_{(i-1)}\alpha_i(\gamma_i + e_i)^{l^-}$

where l^+ and l^- are the length of the parallel paths. One can get a graded quiver for an arbitrary singular surface by gluing along a gentle quiver. The following illustration gives a typical example:

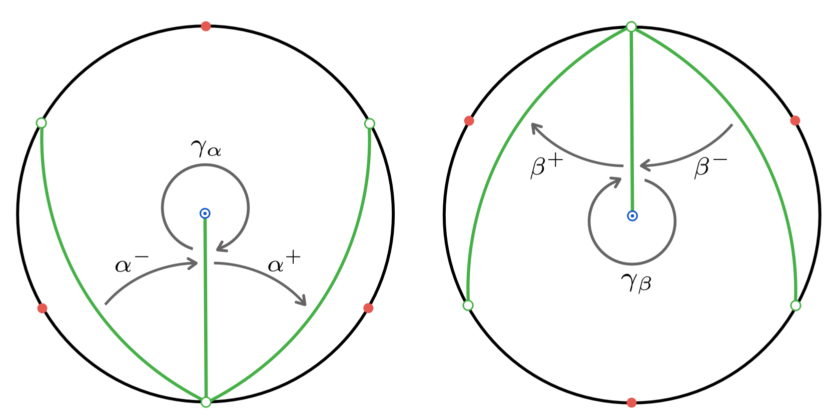


Figure 2. A pinched gentle algebra, and its associated marked surface with conical singularities and admissible dissection.

where the relations are

$$\beta^+\alpha^- = 0,\ \alpha^+\beta^- = 0,\ \alpha^+\gamma = \alpha^+,\ \alpha^-\gamma = \alpha^-,\ \beta^+\gamma = -\beta^+,\ \beta^-\gamma = -\beta^-.$$

References

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