# Combinatorial Auslander–Reiten quivers and their frieze patterns

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## Context

- In [Béd99], Bédard showed how to construct the Auslander–Reiten quiver of a Dynkin quiver from the Coxeter combinatorics of the corresponding Weyl group W.
- Oh and Suh used similar ideas in [OS19] to introduce a purely combinatorial analog of the AR quiver associated with any reduced word in W.
- Building on their work, Fujita and Oh ([FO21]) studied a particular class of such quivers called twisted AR quivers. Their framework has found important applications in the representation theory of quantum algebras.
- They proved that twisted AR quivers satisfy a certain additive property that can be interpreted as a local rule for a frieze pattern.

## Notation for root systems

- Let R be an irreducible root system with Dynkin diagram $\Delta$ of type ADE. For each vertex $i \in \Delta_0$, we denote by $\alpha_i \in$ R the corresponding simple root.
- Denote by W the associated Weyl group and by $s_i$ the simple reflection for $i \in \Delta_0$.
- Let $w_0 \in$ W be the longest element. It induces an involution $i \mapsto i^*$ on $\Delta_0$ determined by $w_0(\alpha_i) = -\alpha_{i^*}$.
- We fix a reduced word $\boldsymbol{i} = (i_1, i_2, \ldots, i_N) \in \Delta_0^N$ of $w_0$. Its extension $\widehat{\boldsymbol{i}} = (i_k)_{k\in\mathbb{Z}}$ is defined by imposing $i_{k+N} = i_k^*$ for all $k \in \mathbb{Z}$.

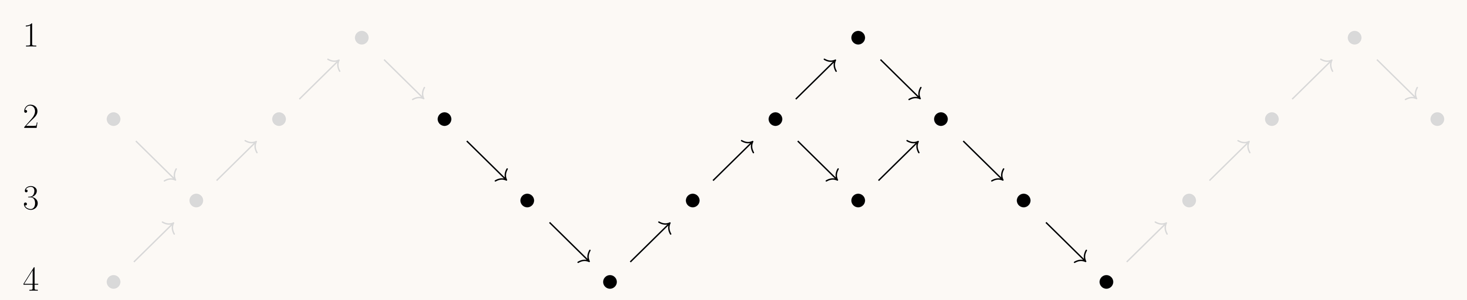
## Combinatorial Auslander–Reiten quivers [OS19]

- The **combinatorial repetition quiver** $\widehat{\Upsilon}_{\boldsymbol{i}}$ is the quiver with vertex set $\mathbb{Z}$ where there is an arrow from $k$ to $l$ if $k > l$, $i_k$ and $i_l$ are adjacent in $\Delta$ and there is no index $l < j < k$ such that $i_j = i_l$ or $i_j = i_k$.
- The **combinatorial Auslander–Reiten quiver** $\Upsilon_{\boldsymbol{i}}$ is the full subquiver of $\widehat{\Upsilon}_{\boldsymbol{i}}$ with vertex set $\{1, 2, \ldots, N\}$.
- We define the **coordinate map** $\rho : (\widehat{\Upsilon}_{\boldsymbol{i}})_0 \to$ R by

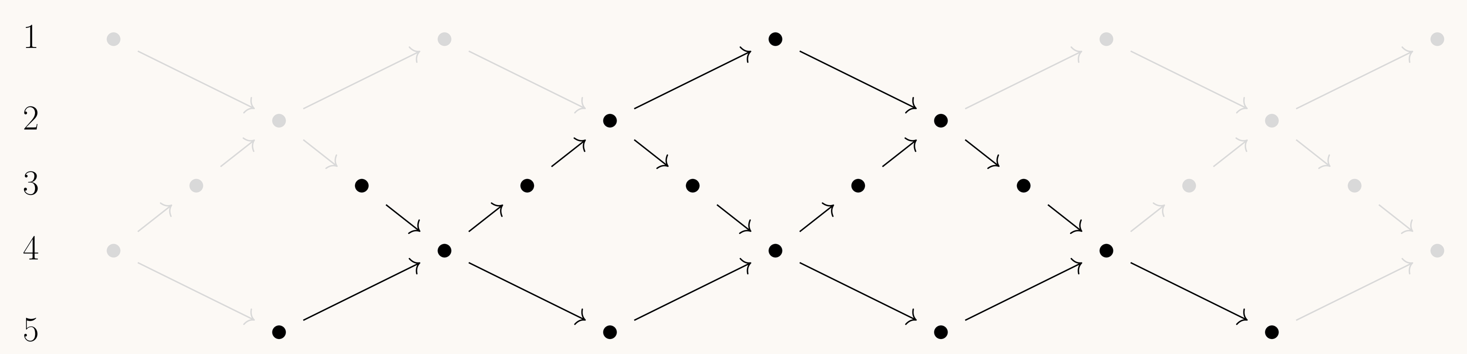
$$\rho(k) = \begin{cases} s_{i_1}s_{i_2}\cdots s_{i_{k-1}}(\alpha_{i_k}) & \text{if } k \geq 1, \\ -s_{i_0}s_{i_{-1}}\cdots s_{i_{k+1}}(\alpha_{i_k}) & \text{if } k \leq 0. \end{cases}$$

## Examples

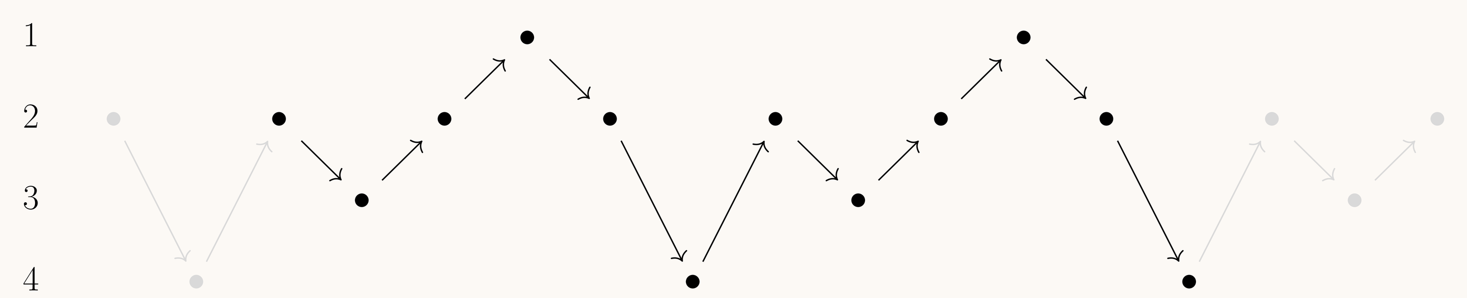
- $\Delta = \mathrm{A}_4$ and $\boldsymbol{i} = (4,3,2,1,3,2,3,4,3,2)$:



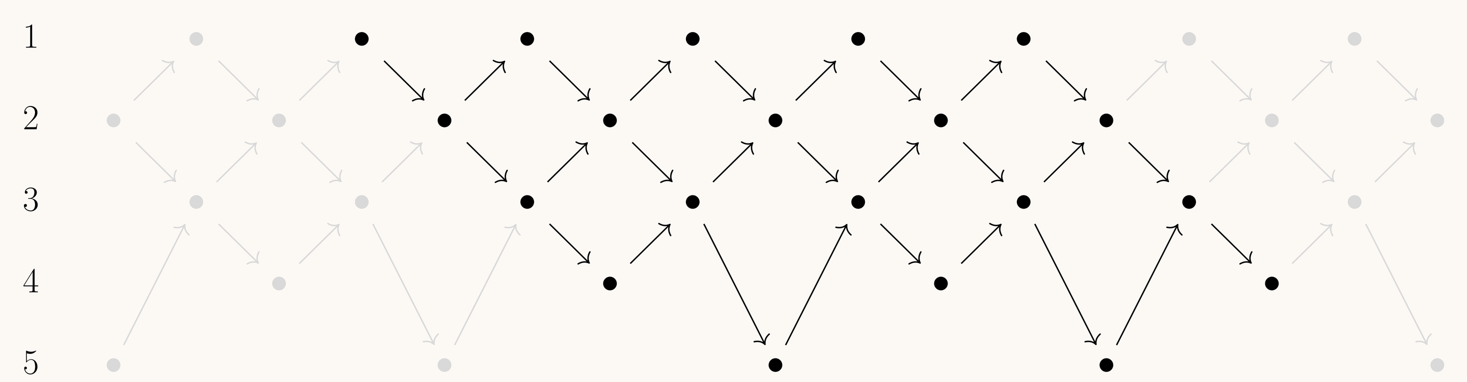

- $\Delta = \mathrm{A}_5$ and $\boldsymbol{i} = (5,4,3,2,5,3,1,4,3,2,5,3,4,3,5)$:




- $\Delta = \mathrm{D}_4$ and $\boldsymbol{i} = (4,2,1,2,3,2,4,2,1,2,3,2)$:




- $\Delta = \mathrm{D}_5$ and $\boldsymbol{i} = (4,3,2,1,5,3,2,1,4,3,2,1,5,3,2,1,4,3,2,1)$:




## Theorem 1: [Béd99; OS19]

*Let $Q$ be a Dynkin quiver of type $\Delta$. Let $\Gamma$ be the Auslander–Reiten quiver of the path algebra of $Q$ over a field. If $\boldsymbol{i} = (i_1, \ldots, i_N)$ is a reduced word for $w_0$ which is also a source sequence for $Q$, then there is an isomorphism of quivers $\varphi : \Gamma \to \Upsilon_{\boldsymbol{i}}$ such that the composition $\rho \circ \varphi_0 : \Gamma_0 \to$ R restricts to the bijection between indecomposable representations of $Q$ and positive roots given by Gabriel's theorem.*

## Meshes in combinatorial AR quivers

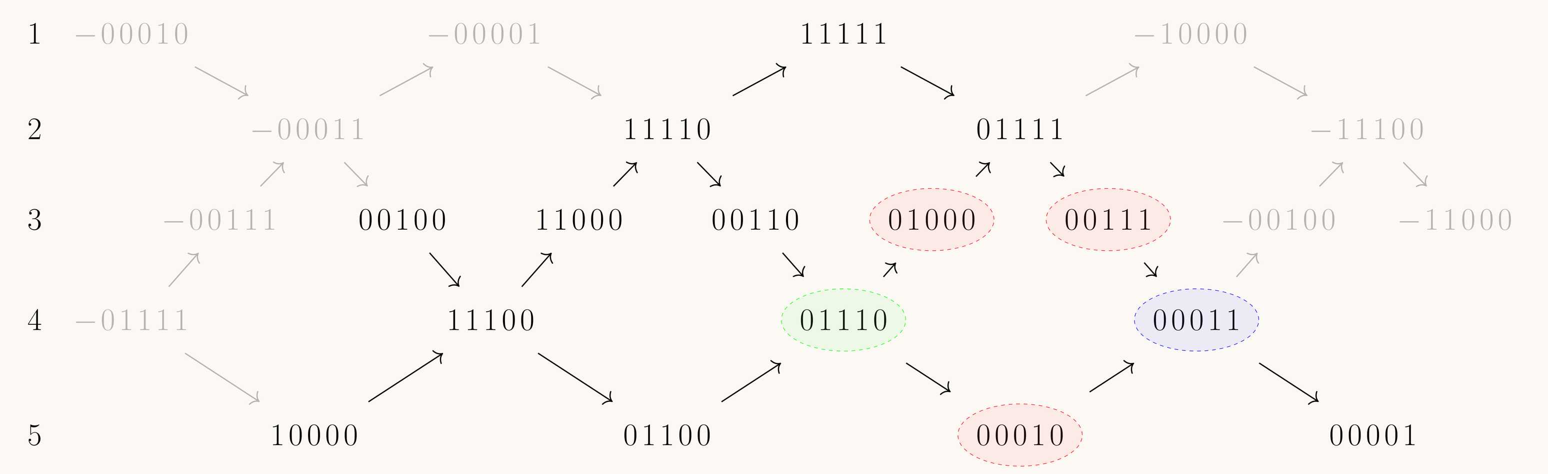
Take a vertex $x \in (\widehat{\Upsilon}_{\boldsymbol{i}})_0 = \mathbb{Z}$ of the combinatorial repetition quiver.

- The **translation** of $x$ is the smallest integer $\tau x \in (\widehat{\Upsilon}_{\boldsymbol{i}})_0$ such that $x < \tau x$ and $i_x = i_{\tau x}$.
- The **set of abutters*** of $x$ is the subset $V_{\boldsymbol{i}}(x) \subset (\widehat{\Upsilon}_{\boldsymbol{i}})_0$ formed by the vertices $y$ such that $x < y < \tau x$ and $i_y$ is adjacent to $i_x$ in $\Delta$.

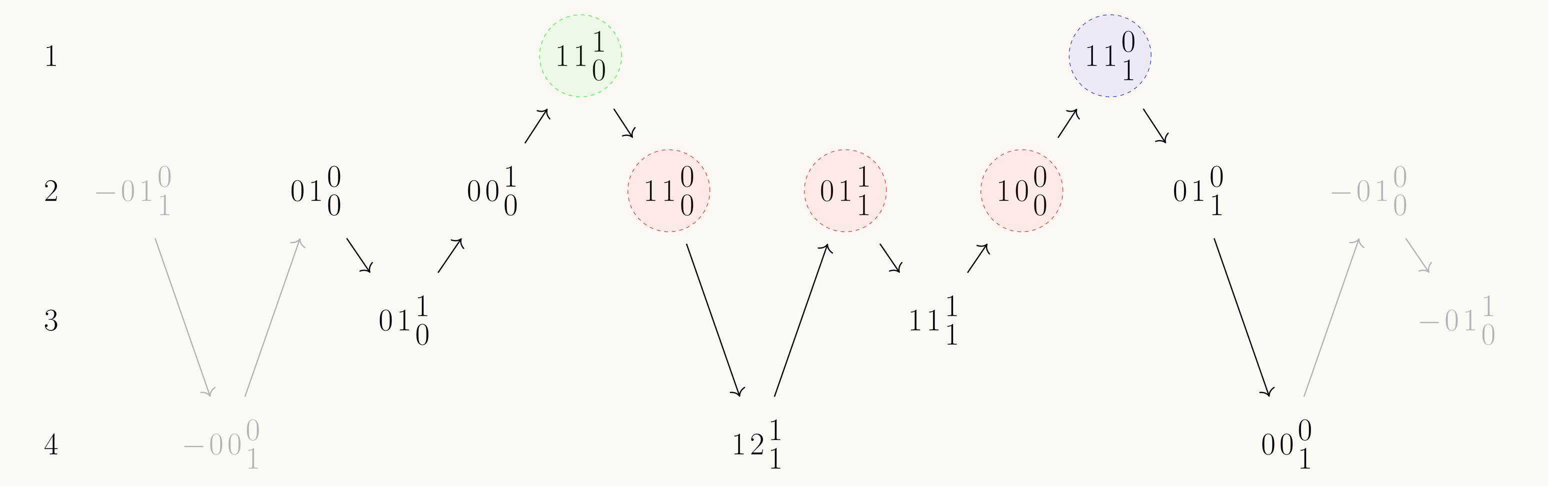
*An abutter is the owner of an adjacent property.

## Example: Coordinate map and meshes

For the previous example in type $\mathrm{A}_5$, over each vertex $x$ of $\widehat{\Upsilon}_{\boldsymbol{i}}$ we write the root $\rho(x)$ given by the coordinate map. The numbers indicate the coefficient of each simple root $\alpha_i$ in $\rho(x)$.




We do the same for the previous example in type $\mathrm{D}_4$:




In both pictures, the translation of the blue vertex is the green one, and the red vertices form the set of abutters of the blue vertex.

## Frieze patterns

An **additive frieze** on $\widehat{\Upsilon}_{\boldsymbol{i}}$ is a function $f : (\widehat{\Upsilon}_{\boldsymbol{i}})_0 \to \mathbb{Z}$ which satisfies the **mesh relations**

$$f(\tau x) + f(x) = \sum_{y \in V_{\boldsymbol{i}}(x)} f(y)$$

for all $x \in (\widehat{\Upsilon}_{\boldsymbol{i}})_0$.

## Theorem 2: [C.] Friezes come from the coordinate map

*For $i \in \Delta_0$, let $\pi_i :$ R $\to \mathbb{Z}$ be the function which returns the coefficient of $\alpha_i$ in a root. Then $\pi_i \circ \rho$ is an additive frieze on $\widehat{\Upsilon}_{\boldsymbol{i}}$. Moreover, these functions form a $\mathbb{Z}$-basis of the space of additive friezes on $\widehat{\Upsilon}_{\boldsymbol{i}}$.*

## Corollary 3: Periodicity

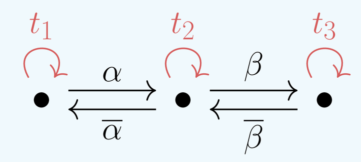
*If $f$ is an additive frieze on $\widehat{\Upsilon}_{\boldsymbol{i}}$, then $f(x + N) = -f(x)$ for all $x \in (\widehat{\Upsilon}_{\boldsymbol{i}})_0$. In particular, $f$ is $2N$-periodic.*

## References


- Claire Amiot, Osamu Iyama, Idun Reiten, and Gordana Todorov. "Preprojective algebras and c-sortable words". *Proc. Lond. Math. Soc. (3)* 104.3 (2012), pp. 513–539.
- Robert Bédard. "On commutation classes of reduced words in Weyl groups". *European J. Combin.* 20.6 (1999), pp. 483–505.
- Ryo Fujita and Se-jin Oh. "Q-data and representation theory of untwisted quantum affine algebras". *Comm. Math. Phys.* 384.2 (2021), pp. 1351–1407.
- Se-jin Oh and Uhi Rinn Suh. "Combinatorial Auslander-Reiten quivers and reduced expressions". *J. Korean Math. Soc.* 56.2 (2019), pp. 353–385.


## Towards a categorification

- Let $Q$ be an orientation of $\Delta$. We define $\Pi$ to be the **2-dimensional Ginzburg dg algebra** associated with $Q$ over some field $K$.
- For example, if $\Delta = \mathrm{A}_3$, then $\Pi$ is the dg path algebra given by




  where black arrows have degree 0 and red arrows have degree $-1$. The differential is determined by $d(t_1 + t_2 + t_3) = [\alpha, \overline{\alpha}] + [\beta, \overline{\beta}]$.
- $\Pi$ is a connective dg algebra whose zeroth cohomology is the preprojective algebra of $Q$. It is smooth and its perfectly valued derived category $\mathrm{pvd}(\Pi)$ is a 2-Calabi-Yau triangulated category.
- Each simple dg module $S_i$ corresponding to $i \in \Delta_0$ is 2-spherical and yields a **spherical twist functor** $T_i : \mathrm{pvd}(\Pi) \to \mathrm{pvd}(\Pi)$, which is an autoequivalence of $\mathrm{pvd}(\Pi)$.
- The map $[S_i] \mapsto \alpha_i$ induces an isomorphism between the Grothendieck group of $\mathrm{pvd}(\Pi)$ and the root lattice of $\Delta$. The action of $T_i$ on the Grothendieck group identifies with the action of the simple reflection $s_i \in$ W.

## Categories associated with reduced words [C.]

- For $k \in \mathbb{Z}$, we define the following object of $\mathrm{pvd}(\Pi)$:

$$M_k^{\boldsymbol{i}} = \begin{cases} T_{i_1}T_{i_2}\cdots T_{i_{k-1}}(S_{i_k}) & \text{if } k \geq 1, \\ \Sigma T_{i_0}^{-1}T_{i_{-1}}^{-1}\cdots T_{i_{k+1}}^{-1}(S_{i_k}) & \text{if } k \leq 0, \end{cases}$$

  where $\Sigma$ denotes the suspension functor of $\mathrm{pvd}(\Pi)$.
- The **repetition category** $\mathcal{R}(\boldsymbol{i})$ is the full additive subcategory of $\mathrm{pvd}(\Pi)$ generated by the indecomposable objects $M_k^{\boldsymbol{i}}$ for $k \in \mathbb{Z}$.
- The **category of representations** $\mathcal{C}(\boldsymbol{i})$ is the full subcategory of $\mathcal{R}(\boldsymbol{i})$ whose objects have cohomology concentrated in degree zero. In particular, we may view its objects as representations of the preprojective algebra.

## Lemma 4: [AIRT12]

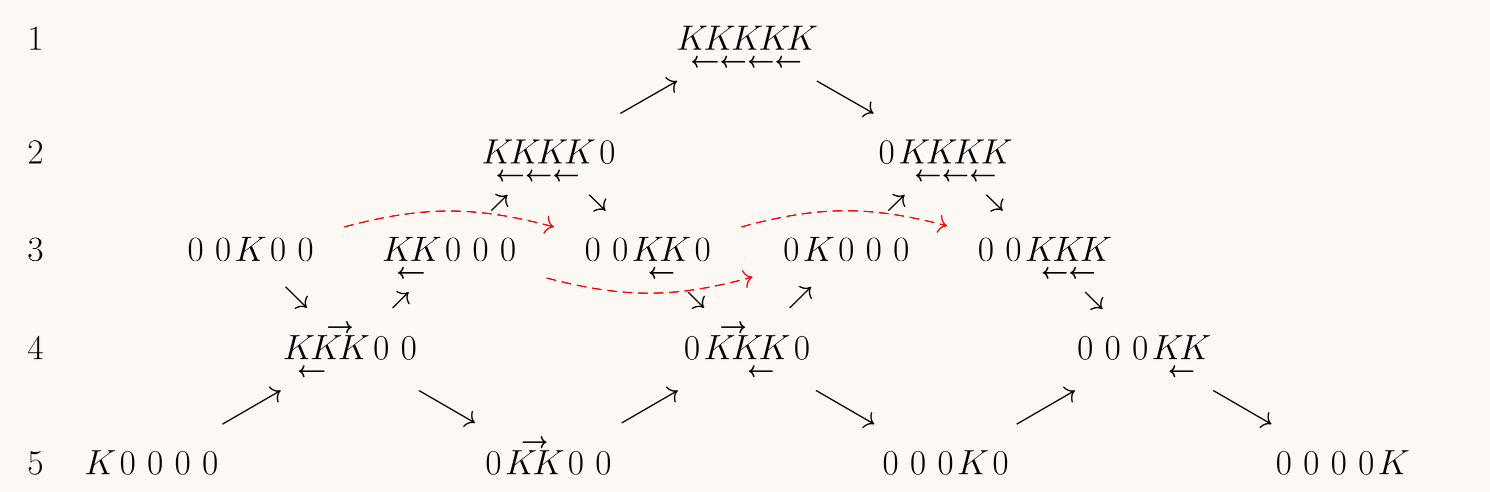
*The indecomposable objects of $\mathcal{C}(\boldsymbol{i})$ are the $M_k^{\boldsymbol{i}}$ for $1 \leq k \leq N$. As representations of the preprojective algebra, they coincide with certain modules called* **layers** *and defined in [AIRT12].*

## Theorem 5: [C.] Categorification of combinatorial AR quivers

- *The combinatorial AR quiver $\Upsilon_{\boldsymbol{i}}$ is isomorphic to the quiver obtained from the Gabriel quiver of $\mathcal{C}(\boldsymbol{i})$ by removing all arrows parallel to paths of length at least two.*
- *We have the analogous statement for the combinatorial repetition quiver $\widehat{\Upsilon}_{\boldsymbol{i}}$ and $\mathcal{R}(\boldsymbol{i})$.*

## Example

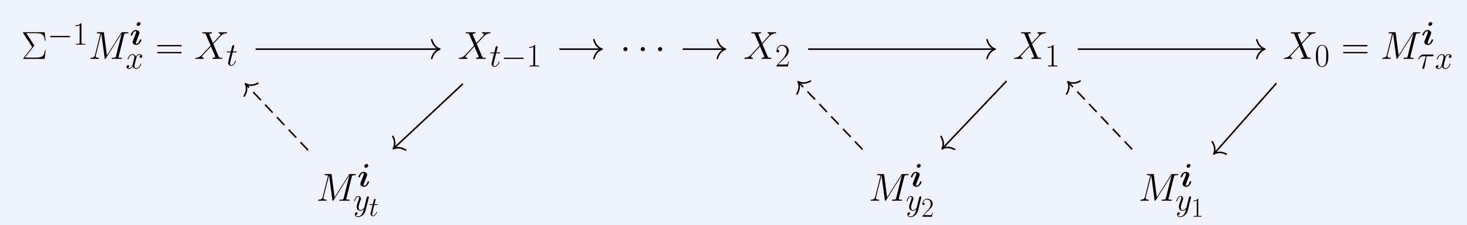
For the previous example in type $\mathrm{A}_5$, the Gabriel quiver of $\mathcal{C}(\boldsymbol{i})$ is the following:




Each indecomposable object is viewed as a representation of the preprojective algebra of type $\mathrm{A}_5$.

## Theorem 6: [C.] Categorification of the mesh relations

*Given a vertex $x \in (\widehat{\Upsilon}_{\boldsymbol{i}})_0$, choose an ordering $y_1, \ldots, y_t$ of the set of abutters $V_{\boldsymbol{i}}(x)$ such that $k \leq l$ whenever there is a path from $y_k$ to $y_l$ in $\widehat{\Upsilon}_{\boldsymbol{i}}$. Then there are indecomposable objects $X_1, X_2, \ldots, X_{t-1}$ in $\mathcal{R}(\boldsymbol{i})$ and a diagram of the form*




*where the triangles above are distinguished triangles in $\mathrm{pvd}(\Pi)$.*