Combinatorial Auslander–Reiten quivers and their frieze patterns

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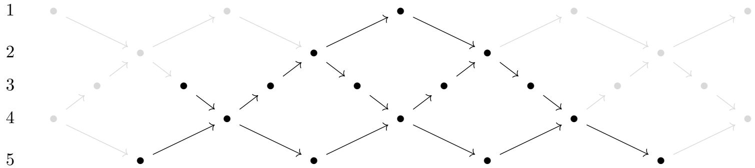
Combinatorial Auslander–Reiten quivers

- Let Δ be a Dynkin diagram of type ADE.
- Bédard (1999) studied the relation between the AR quiver of a Dynkin quiver of type Δ and certain reduced words for the longest element $w_0 \in \mathsf{W}$.
- Oh and Suh (2019) defined the combinatorial AR quiver $\Upsilon_{\boldsymbol{i}}$ associated with any reduced word $\boldsymbol{i}$ for w_0. It can be naturally extended to a repetition quiver $\widehat{\Upsilon}_{\boldsymbol{i}}$.

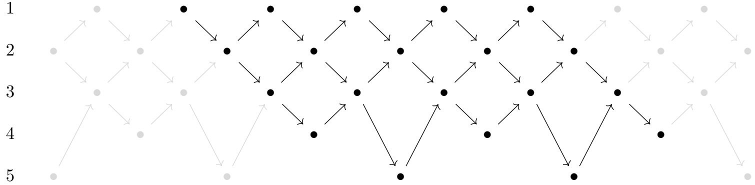
Examples

In type A_5:



Examples

In type D_5:



Additive friezes

- Each vertex x in $\widehat{\Upsilon}_{\boldsymbol{i}}$ has a translation τx.
- The set $V_{\boldsymbol{i}}(x)$ of "close neighbors" between x and τx forms the analog of a mesh in an AR quiver.

Definition

An additive frieze on $\widehat{\Upsilon}_{\boldsymbol{i}}$ is a function $f : (\widehat{\Upsilon}_{\boldsymbol{i}})_0 \to \mathbb{Z}$ satisfying

$$f(\tau x) + f(x) = \sum_{y \in V_{\boldsymbol{i}}(x)} f(y)$$

for all $x \in (\widehat{\Upsilon}_{\boldsymbol{i}})_0$.

Example

For the previous example in type A_5, we have the following additive frieze:

$$
\begin{array}{c|ccccccccccccccccc}
1 & -1 & & & & -1 & & & & 5 & & & & -1 & & & & -2 \\
2 & & & -2 & & & & 4 & & & & 4 & & & & -3 & & \\
3 & & -3 & & 1 & & 2 & & 2 & & 1 & & 3 & & -1 & & -2 & \\
4 & -4 & & & & 3 & & & & 3 & & & & 2 & & & & -4 \\
5 & & & 1 & & & & 2 & & & & 1 & & & & 1 & &
\end{array}
$$

Finding all friezes

There is a coordinate map from the set of vertices of $\widehat{\Upsilon}_{\boldsymbol{i}}$ to the root system of type Δ.

Theorem

The coordinate map satisfies the mesh relations. It gives rise to all additive friezes on $\widehat{\Upsilon}_{\boldsymbol{i}}$.

This was known only for twisted AR quivers.

Corollary

All additive friezes are periodic and have similar symmetries as in the classical case.

Categorification

- Let Π be the derived preprojective algebra of type Δ.
- We define a certain additive subcategory $\mathcal{R}(\boldsymbol{i})$ of $\mathrm{pvd}(\Pi)$.

Theorem

The combinatorial repetition quiver $\widehat{\Upsilon}_{\boldsymbol{i}}$ can be obtained from the Gabriel quiver of $\mathcal{R}(\boldsymbol{i})$ after removing certain arrows.

Theorem

The mesh relations satisfied by the coordinate map are categorified by a certain sequence of distinguished triangles in $\mathrm{pvd}(\Pi)$ involving objects in $\mathcal{R}(\boldsymbol{i})$.

For the previous example in type A_5, we can depict the “heart” of $\mathcal{R}(\boldsymbol{i})$ as follows:

$$
\begin{tikzcd}[column sep=tiny]
1 & & & & & & & \underset{\leftarrow\leftarrow\leftarrow\leftarrow}{KKKKK} \arrow[drr] & & & & & & \\
2 & & & & & \underset{\leftarrow\leftarrow\leftarrow}{KKKK}0 \arrow[urr] \arrow[dr] & & & & 0\underset{\leftarrow\leftarrow\leftarrow}{KKKK} \arrow[dr] & & & & \\
3 & & 00K00 \arrow[dr] \arrow[rrrr, dashed, red, bend left] & & \underset{\leftarrow}{KK}000 \arrow[ur] \arrow[rrrr, dashed, red, bend right] & & 00\underset{\leftarrow}{KK}0 \arrow[dr] \arrow[rrrr, dashed, red, bend left] & & 0K000 \arrow[ur] & & 00\underset{\leftarrow\leftarrow}{KKK} \arrow[dr] & & & \\
4 & & & \overset{\rightarrow}{\underset{\leftarrow}{KKK}}00 \arrow[ur] \arrow[drr] & & & & 0\overset{\rightarrow}{\underset{\leftarrow}{KKK}}0 \arrow[ur] \arrow[drr] & & & & 000\underset{\leftarrow}{KK} \arrow[drr] & & \\
5 & K0000 \arrow[urr] & & & & 0\overset{\rightarrow}{KK}00 \arrow[urr] & & & & 000K0 \arrow[urr] & & & & 0000K
\end{tikzcd}
$$