Dg enhanced orbit categories and applications

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Cluster categorier as the orbit categories

- Let A be a finite-dimensional $\mathbf{k}$-algebra with $\operatorname{gldim} A < \infty$ and $m \geq 2$ an integer.
- In the hereditary case and $m = 2$, Buan-Marsh-Reineke-Reiten-Todorov (2005) defined the cluster category as the orbit category
$$\mathcal{C}_2(A) := \mathcal{D}^b(\operatorname{mod} A)/\tau^{-1} \circ [1],$$
where τ is the AR-translation functor. Keller (2005) shown that $\mathcal{C}_2(A)$ does carry a canonical triangulated structure which is obtained by using dg enhancements.
- In general, $\operatorname{per} A = \mathcal{D}^b(A)$ admits a Serre functor $\mathbb{S} := - \overset{\mathbf{L}}{\otimes}_A DA$. Set $\Sigma_m := \mathbb{S} \circ [-m]$, which is an autoequivalence of $\operatorname{per} A$. The orbit category $\operatorname{per} A/\Sigma_m$ has a triangulated hull (Keller)
$$\mathcal{C}_m(A) := \langle A \rangle_B / \operatorname{per} B,$$
where B is the dg algebra $A \oplus DA[-m-1]$. Moreover, the m-cluster category $\mathcal{C}_m(A)$ of A coincides with $\operatorname{per} A/\Sigma_m$ when A is hereditary.

Construction of dg orbit categories

Let $\mathcal{A}$ be a dg category and $F \in \operatorname{rep}_{\mathrm{dg}}(\mathcal{A}, \mathcal{A})$ a dg bimodule. The *left lax quotient* $\mathcal{A}/_{\mathrm{ll}}F^{\mathbb{N}}$ *of* $\mathcal{A}$ *by* F is the dg category whose

- objects are the same as the objects of $\mathcal{A}$, and
- morphisms are given by $\mathcal{A}/_{\mathrm{ll}}F^{\mathbb{N}}(X, Y) = \oplus_{p \in \mathbb{N}} \mathcal{A}(X, F^p Y)$.

The canonical dg functor $Q_{\mathbb{N}} : \mathcal{A} \to \mathcal{A}/_{\mathrm{ll}}F^{\mathbb{N}}$ acts on

- objects: it sends $X \in \operatorname{obj}(\mathcal{A})$ to $X \in \operatorname{obj}(\mathcal{A}/_{\mathrm{ll}}F^{\mathbb{N}})$, and
- morphisms: it sends $f : X \to Y$ to $f \in \mathcal{A}(X, Y) \subseteq \oplus_{p \in \mathbb{N}} \mathcal{A}(X, F^p Y)$.

The canonical morphism of dg functors $q : Q_{\mathbb{N}} F \to Q_{\mathbb{N}}$ acts on the objects of $\mathcal{A}$ as
$$qX := \mathrm{id}_{FX} \in \mathcal{A}(FX, FX) \subseteq \oplus_{p \in \mathbb{N}} \mathcal{A}(FX, F^{p+1}X).$$

Definition

The *dg orbit category* $\mathcal{A}/F^{\mathbb{Z}}$ is defined to be the dg localization
$$(\mathcal{A}/_{\mathrm{ll}}F^{\mathbb{N}})[q^{-1}]$$
of $\mathcal{A}/_{\mathrm{ll}}F^{\mathbb{N}}$ with respect to the morphisms $qX : Q_{\mathbb{N}}FX \to Q_{\mathbb{N}}X$ for any $X \in \operatorname{obj}(\mathcal{A})$.

We use the *$\mathbb{Z}$-equivariant category* $\mathbb{Z}\text{-}\operatorname{Eq}(\mathcal{A}, F, \mathcal{B})$ to describe the universal property of dg orbit category. It consists of $\mathbb{Z}$-equivariant functors from $(\mathcal{A}, F)$ to $(\mathcal{B}, \mathrm{id}_{\mathcal{B}})$ i.e. (G, γ), where $G \in \operatorname{rep}_{\mathrm{dg}}(\mathcal{A}, \mathcal{B})$ and $\gamma : GF \to G$ s.t. γX is an isomorphism in $H^0(\mathcal{B}), \forall X \in \operatorname{obj}(\mathcal{A})$. The pretriangulated hull of the dg orbit category satisfies the following universal property.

Theorem

Let $\mathcal{B}$ *be a pretriangulated dg category. Then* $\mathcal{A} \to \operatorname{pretr}(\mathcal{A}/F^{\mathbb{Z}})$ *induces an isomorphism*
$$\operatorname{rep}_{\mathrm{dg}}(\operatorname{pretr}(\mathcal{A}/F^{\mathbb{Z}}), \mathcal{B}) \xrightarrow{\sim} \mathbb{Z}\text{-}\operatorname{Eq}(\mathcal{A}, F, \mathcal{B}) \tag{1}$$
in Hqe.

Thus, we can construct the triangulated orbit category of a triangulated category in a canonical way.

Definition

Let $\mathcal{T}$ be a triangulated category endowed with a dg enhancement $H^0(\mathcal{A}) \xrightarrow{\sim} \mathcal{T}$ and $F \in \operatorname{rep}_{\mathrm{dg}}(\mathcal{A}, \mathcal{A})$ be a dg bimodule. If the induced functor $H^0(F) : H^0(\mathcal{A}) \to H^0(\mathcal{A})$ is an equivalence, then the *triangulated orbit category* of $\mathcal{T}$ wrt $\mathcal{A}$ is defined as
$$H^0(\operatorname{pretr}(\mathcal{A}/F^{\mathbb{Z}})). \tag{2}$$
Moreover, we have that taking dg orbit commutes with taking dg quotient.

Proposition

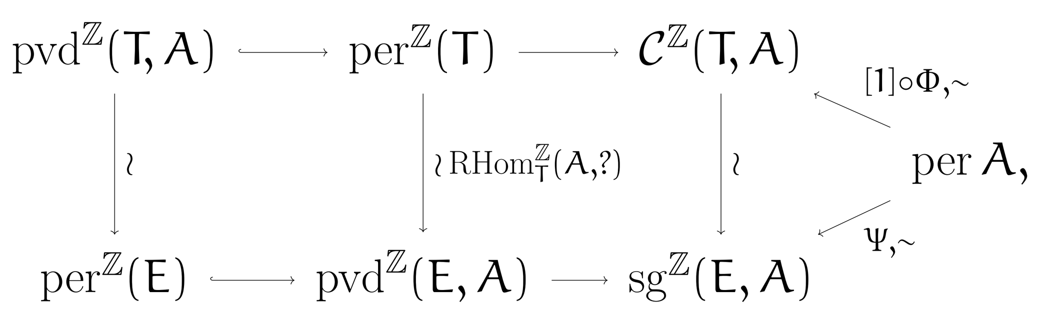
Suppose $\mathcal{N} \subseteq \mathcal{A}$ *is a full dg subcategory such that* $H^0(\mathcal{N})$ *is stable under* $H^0(F)$, *so that* F *induces dg bimodules* $F_{\mathcal{N}} \in \operatorname{rep}_{\mathrm{dg}}(\mathcal{N}, \mathcal{N})$ *and* $F_{\mathcal{A}/\mathcal{N}} \in \operatorname{rep}_{\mathrm{dg}}(\mathcal{A}/\mathcal{N}, \mathcal{A}/\mathcal{N})$. *We have a canonical isomorphism in* Hqe
$$\operatorname{pretr}((\mathcal{A}/\mathcal{N})/F^{\mathbb{Z}}_{\mathcal{A}/\mathcal{N}}) \simeq \operatorname{pretr}(\mathcal{A}/F^{\mathbb{Z}})/\operatorname{pretr}(\mathcal{N}/F^{\mathbb{Z}}_{\mathcal{N}}) \tag{3}$$
and a short exact sequence of triangulated categories
$$0 \to (\mathcal{N}/F^{\mathbb{Z}}_{\mathcal{N}})^{\mathrm{tr}} \to (\mathcal{A}/F^{\mathbb{Z}})^{\mathrm{tr}} \to ((\mathcal{A}/\mathcal{N})/F^{\mathbb{Z}}_{\mathcal{A}/\mathcal{N}})^{\mathrm{tr}} \to 0, \tag{4}$$
where $(-)^{\mathrm{tr}}$ *denotes the functor* $H^0 \circ \operatorname{pretr}$.

Cluster categories versus singularity categories

Let A be a dg algebra and $X \in \mathcal{D}(A^e)$ an invertible dg bimodule with inverse Y, that is, there are isomorphisms
$$X \overset{\mathbf{L}}{\otimes}_A Y \simeq A \text{ and } Y \overset{\mathbf{L}}{\otimes}_A X \simeq A$$
in $\mathcal{D}(A^e)$. Write $\bar{X}$ for a cofibrant resolution of $X[\mathbb{X} - 1]$, where $[\mathbb{X} - 1]$ indicates the shift by bidegree $(-1, 1)$. We let
$$T := T_A(\bar{X}) = \oplus_{p \geq 0} \bar{X}^{\otimes_A^p}$$
be the *differential bigraded tensor algebra of* $\bar{X}$ *over* A and
$$E := A \oplus Y[-\mathbb{X}]$$
the *differential bigraded trivial extension algebra of* $Y[-\mathbb{X}]$ over A. We define the *enlarged cluster category* of A with respect to X as the Verdier quotient
$$\mathcal{C}^{\mathbb{Z}}(T, A) := \operatorname{per}^{\mathbb{Z}}(T)/\operatorname{pvd}^{\mathbb{Z}}(T, A); \tag{5}$$
and the *shrunk singularity category* of A as the Verdier quotient
$$\operatorname{sg}^{\mathbb{Z}}(E, A) := \operatorname{pvd}^{\mathbb{Z}}(E, A)/\operatorname{per}^{\mathbb{Z}}(E). \tag{6}$$

Theorem

The adjoint pair
$$\operatorname{per}^{\mathbb{Z}}(T) \underset{? \overset{\mathbf{L}}{\otimes}_E A}{\overset{\operatorname{RHom}^{\mathbb{Z}}_T(A, ?)}{\rightleftarrows}} \operatorname{pvd}^{\mathbb{Z}}(E, A)$$
induces the following commutative diagram

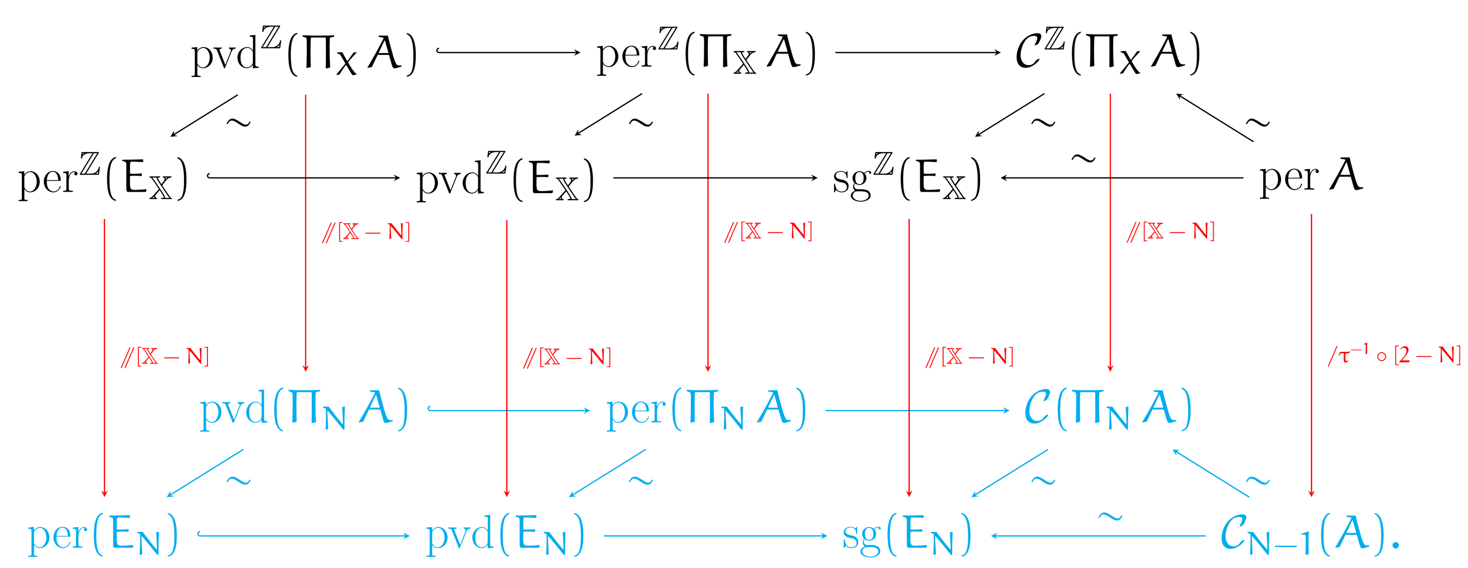


(7)

where we have $[\mathbb{X}] \circ [1] \circ \Phi \simeq [1] \circ \Phi \circ (? \overset{\mathbf{L}}{\otimes}_A Y[1]), [\mathbb{X}] \circ \Psi \simeq \Psi \circ (? \overset{\mathbf{L}}{\otimes}_A Y[1])$ *and all the other functors commute with* $[\mathbb{X}]$.

Application: N-reductions as taking orbits

We consider the case when A is a smooth, proper and connective dg algebra, $X = A^{\vee}$ and $Y = DA$, where $A^{\vee}$ is a cofibrant replacement of the dg bimodule $\operatorname{RHom}_{A^e}(A, A^e)$. Then $T = \Pi_{\mathbb{X}} A$ is the $\mathbb{X}$-Calabi-Yau completion, which is the dbg algebra
$$\Pi_{\mathbb{X}} A = A \oplus \Theta \oplus (\Theta \otimes_A \Theta) \oplus \cdots$$
for $\Theta = A^{\vee}[\mathbb{X} - 1]$, and the trivial extension $E_{\mathbb{X}} = A \oplus DA[-\mathbb{X}]$ of A respectively. Moreover, the enlarged cluster category reduces to the ∞-cluster category
$$\mathcal{C}^{\mathbb{Z}}(\Pi_{\mathbb{X}} A) := \operatorname{per}^{\mathbb{Z}}(\Pi_{\mathbb{X}} A)/\operatorname{pvd}^{\mathbb{Z}}(\Pi_{\mathbb{X}} A),$$
and the shrunk singularity category becomes
$$\operatorname{sg}^{\mathbb{Z}}(E_{\mathbb{X}}) := \operatorname{pvd}^{\mathbb{Z}}(E_{\mathbb{X}})/\operatorname{per}^{\mathbb{Z}}(E_{\mathbb{X}}).$$
We have a canonical isomorphism of vector spaces
$$\oplus_{(a,b) \in \mathbb{Z}^2, a+bN=p} (\Pi_{\mathbb{X}} A)^a_b \to (\Pi_N A)^p.$$
If we denote the dg algebra $E_N := A \oplus DA[-N]$, then, on the level of differential (bi)graded algebras, we have the canonical projections
$$\pi_N \colon \Pi_{\mathbb{X}} A \to \Pi_N A \text{ and } \pi_N \colon E_{\mathbb{X}} \to E_N \tag{8}$$
collapsing the double degree $(a, b) \in \mathbb{Z} \oplus \mathbb{Z}\mathbb{X}$ into $a + bN \in \mathbb{Z}$. They induce functors
$$\pi_N \colon \operatorname{per}^{\mathbb{Z}}(\Pi_{\mathbb{X}} A) \to \operatorname{per}(\Pi_N A) \text{ and } \pi_N \colon \operatorname{pvd}^{\mathbb{Z}}(E_{\mathbb{X}}) \to \operatorname{pvd}(E_N),$$
and restrict to the responding perfected valued subcategories. As a consequence, we have the following commutative diagram between short exact sequences of triangulated categories:



Main References

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