Dg enhanced orbit categories and applications

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Dg orbit categories

Let $\mathcal{A}$ be a dg category and $F \in \mathrm{rep}_{dg}(\mathcal{A}, \mathcal{A})$ a dg bimodule.
The left lax quotient $\mathcal{A}/_{ll}F^{\mathbb{N}}$ of $\mathcal{A}$ by F is the dg category whose

- objects are the same as the objects of $\mathcal{A}$, and
- morphisms are given by $\mathcal{A}/_{ll}F^{\mathbb{N}}(X, Y) = \bigoplus_{p \in \mathbb{N}} \mathcal{A}(X, F^p Y)$.

The canonical dg functor $Q_{\mathbb{N}} : \mathcal{A} \to \mathcal{A}/_{ll}F^{\mathbb{N}}$ acts on

- objects: it sends $X \in \mathrm{obj}(\mathcal{A})$ to $X \in \mathrm{obj}(\mathcal{A}/_{ll}F^{\mathbb{N}})$, and
- morphisms: it sends $f : X \to Y$ to $f \in \mathcal{A}(X, Y) \subseteq \oplus_{p \in \mathbb{N}} \mathcal{A}(X, F^p Y)$.

The canonical morphism of dg functors $q : Q_{\mathbb{N}} F \to Q_{\mathbb{N}}$ acts on the objects of $\mathcal{A}$ as

$$qX \coloneqq \mathrm{id}_{FX} \in \mathcal{A}(FX, FX) \subseteq \bigoplus_{p \in \mathbb{N}} \mathcal{A}(FX, F^{p+1}X).$$

Definition

The dg orbit category $\mathcal{A}/F^{\mathbb{Z}}$ is defined to be the dg localization $(\mathcal{A}/_{ll}F^{\mathbb{N}})[q^{-1}]$ of $\mathcal{A}/_{ll}F^{\mathbb{N}}$ wrt the morphisms $qX : Q_{\mathbb{N}}FX \to Q_{\mathbb{N}}X$ for any $X \in \mathrm{obj}(\mathcal{A})$.

Universal property

The $\mathbb{Z}$-equivariant category $\mathbb{Z}\text{-}\mathrm{Eq}(\mathcal{A}, F, \mathcal{B})$ consists of $\mathbb{Z}$-equivariant functors from $(\mathcal{A}, F)$ to $(\mathcal{B}, \mathrm{id}_{\mathcal{B}})$ i.e. (G, γ), where $G \in \mathrm{rep}_{dg}(\mathcal{A}, \mathcal{B})$ and $\gamma : GF \to G$ s.t. γX is an isom in $H^0(\mathcal{B}), \forall X \in \mathrm{obj}(\mathcal{A})$.

Theorem

Let $\mathcal{B}$ be a pretriangulated dg category. Then $\mathcal{A} \to \mathrm{pretr}(\mathcal{A}/F^{\mathbb{Z}})$ induces an isomorphism

$$\mathrm{rep}_{dg}(\mathrm{pretr}(\mathcal{A}/F^{\mathbb{Z}}), \mathcal{B}) \xrightarrow{\sim} \mathbb{Z}\text{-}\mathrm{Eq}(\mathcal{A}, F, \mathcal{B}). \tag{1}$$

in Hqe.

Definition

Let $\mathcal{T}$ be a triangulated category endowed with a dg enhancement $H^0(\mathcal{A}) \xrightarrow{\sim} \mathcal{T}$ and $F \in \mathrm{rep}_{dg}(\mathcal{A}, \mathcal{A})$ be a dg bimodule. If the induced functor $H^0(F) : H^0(\mathcal{A}) \to H^0(\mathcal{A})$ is an equivalence, then the triangulated orbit category of $\mathcal{T}$ wrt $\mathcal{A}$ is defined as

$$H^0(\mathrm{pretr}(\mathcal{A}/F^{\mathbb{Z}})).$$

Commutation with the dg quotient

Corollary

Suppose $\mathcal{N} \subseteq \mathcal{A}$ is a full dg subcategory such that $H^0(\mathcal{N})$ is stable under $H^0(F)$, so that F induces dg bimodules $F_{\mathcal{N}} \in \mathrm{rep}_{dg}(\mathcal{N}, \mathcal{N})$ and $F_{\mathcal{A}/\mathcal{N}} \in \mathrm{rep}_{dg}(\mathcal{A}/\mathcal{N}, \mathcal{A}/\mathcal{N})$. We have a canonical isomorphism in Hqe

$$\mathrm{pretr}((\mathcal{A}/\mathcal{N})/F^{\mathbb{Z}}_{\mathcal{A}/\mathcal{N}}) \simeq \mathrm{pretr}(\mathcal{A}/F^{\mathbb{Z}})/\,\mathrm{pretr}(\mathcal{N}/F^{\mathbb{Z}}_{\mathcal{N}}). \tag{2}$$

Moreover, we have a short exact sequence of triangulated categories

$$0 \to (\mathcal{N}/F^{\mathbb{Z}}_{\mathcal{N}})^{tr} \to (\mathcal{A}/F^{\mathbb{Z}})^{tr} \to ((\mathcal{A}/\mathcal{N})/F^{\mathbb{Z}}_{\mathcal{A}/\mathcal{N}})^{tr} \to 0, \tag{3}$$

where $(-)^{tr}$ denotes the functor $H^0 \circ \mathrm{pretr}$.

Relative Koszul duality

Theorem

The adjoint pair

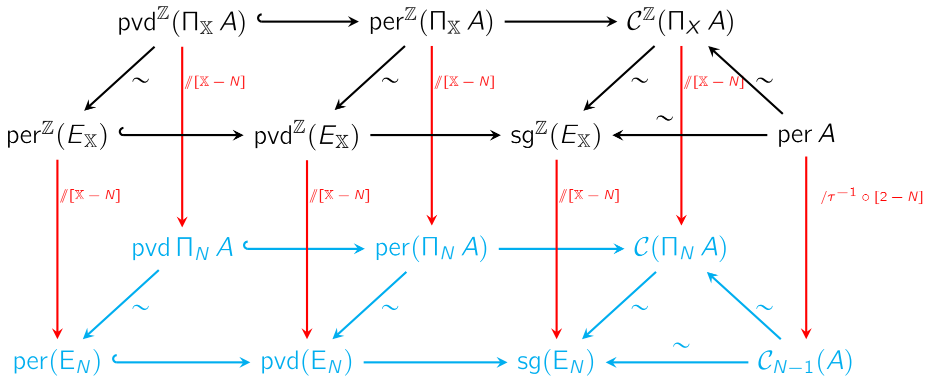
$$\mathrm{per}^{\mathbb{Z}}(\mathsf{T}) \underset{?\overset{L}{\otimes}_{\mathsf{E}} A}{\overset{\mathrm{RHom}^{\mathbb{Z}}_{\mathsf{T}}(A,?)}{\rightleftarrows}} \mathrm{pvd}^{\mathbb{Z}}(\mathsf{E}, A). \tag{4}$$

induces the following commutative diagram

$$\begin{array}{ccccccc}
\mathrm{pvd}^{\mathbb{Z}}(\mathsf{T}, A) & \hookrightarrow & \mathrm{per}^{\mathbb{Z}}(\mathsf{T}) & \longrightarrow & \mathcal{C}^{\mathbb{Z}}(\mathsf{T}, A) & \xleftarrow{[1]\circ\Phi,\sim} & \\
\downarrow \wr & & \downarrow \wr\, \mathrm{RHom}^{\mathbb{Z}}_{\mathsf{T}}(A,?) & & \downarrow \wr & & \mathrm{per}\, A, \\
\mathrm{per}^{\mathbb{Z}}(\mathsf{E}) & \hookrightarrow & \mathrm{pvd}^{\mathbb{Z}}(\mathsf{E}, A) & \longrightarrow & \mathrm{sg}^{\mathbb{Z}}(\mathsf{E}, A) & \xleftarrow{\Psi,\sim} &
\end{array} \tag{5}$$

where "all" functors commute with $[\mathbb{X}]$.

Application



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Thank you!