Cluster structures on schemes of bands

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Based on a Joint work with Bernard Leclerc : [FL25]

Overview

Complexified Grothendieck rings of certain monoidal subcategories of representations of quantum affine algebras, their Borel subalgebras, and shifted quantum affine algebras admit three notable cluster structures introduced by Hernandez-Leclerc [HL10], [HL16] and Geiss-Hernandez-Leclerc [GHL24]. Schemes of bands provide a geometric realisation of these cluster structures. For those interested in connections between bands and representation theory, we recommend to have a look at [FL25, Section 8].

We fix some notation:

- K : algebraically closed field, $\mathrm{char}(K) = 0$.
- G : simple, simply connected, simply laced algebraic group over K.
- T : maximal torus of T.
- B, B^- : Borel subgroups of G such that $B \cap B^- = T$.
- U, U^-: unipotent radicals of B and B^-.
- W : Weyl group of G with respect to T.
- w_0 : longest element of W.
- c : a Coxeter element of W.
- $\widetilde{c} := w_0 c^{-1} w_0$.
- $\overline{c}$: a representative of c in the normaliser of T.
- $U(c^{-1}) := U \cap (cU^- c^{-1})$.
- I : set of vertices of the Dynkin diagram of G.
- $\varpi_i \, (i \in I)$: fundamental weights.
- $\Delta_{v\varpi_i, w\varpi_i} (v, w \in W, i \in I)$: Fomin-Zelevinsky generalised minors; certain regular functions on G.

Schemes of bands and their regular functions

Definition: A (G, c)-band over K is a sequence $b = (g(s))_{s\in\mathbb{Z}}$ of elements of G such that for any $s \in \mathbb{Z}$ we have $g(s)g(s+1)^{-1} \in U(c^{-1})\overline{c}$.

Theorem: The (G, c)-bands over K are the K-rational points of an infinite dimensional affine scheme $B(G, c)$. The ring $R(G, c)$ of regular functions on $B(G, c)$ is a unique factorisation domain.

The scheme $B(G, c)$ represents the functor $\mathrm{K}-\mathrm{Alg} \longrightarrow \mathrm{Set}$ obtained by replacing the field K with a general K-algebra R in the definition of (G, c)-bands over K.

The scheme $B(G, c)$ admits some distinguished regular functions that we will now describe. Let $s \in \mathbb{Z}$ and $k \in \mathbb{Z}_{>0}$. Consider the morphisms

$$\begin{aligned} \pi_s : B(G,c) &\longrightarrow G & \tau_{s,k} : B(G,c) &\longrightarrow G \\ b = ((g(t))_{t\in\mathbb{Z}} &\longmapsto g(s) & b = ((g(t))_{t\in\mathbb{Z}} &\longmapsto g(s)g(s+k)^{-1}, \end{aligned} \tag{1}$$

and define

$$\Delta^{(s)}_{v\varpi_i, w\varpi_i} := {\pi_s}^*(\Delta_{v\varpi_i, w\varpi_i}), \quad \theta^{(s)}_{i,k} := \tau^*_{s,k}(\Delta_{\varpi_i,\varpi_i}), \quad (i \in I, \ v, w \in W). \tag{2}$$

Proposition: The pullback homomorphisms $\pi_s^* : K[G] \longrightarrow R(G, c)$ are injective. Moreover:

1. The K-algebra $R(G, c)$ is generated by the elements
$$\Delta^{(s)}_{v\varpi_i, w\varpi_i} \ (i \in I, \ s \in \mathbb{Z}, \ v, w \in W).$$
2. The algebra $R(G, c)$ is a polynomial ring in in the elements $\theta^{(s)}_{i,1}$ $(i \in I, s \in \mathbb{Z})$ with coefficients in $\pi_0^*(K[G])$.

Definition: The group G acts on $B(G, c)$ by the formula $(g(s))_{s\in\mathbb{Z}} \cdot g := (g(s)g)_{s\in\mathbb{Z}}$.

An explicit example: $(SL(n), c_{st})$-bands

Assume that $G = SL(n)$ and $c = c_{st} := s_1 \cdots s_{n-1}$. Then

$$U(c_{st}^{-1})\overline{c_{st}} = \left\{ \begin{pmatrix} a_1 & a_2 & \cdots & a_{n-1} & (-1)^{n-1} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} : \ a_1, \ldots, a_{n-1} \in K \right\}.$$

Thus, two elements $g, h \in \mathrm{SL}(n)$ satisfy $gh^{-1} \in U(c_{st}^{-1})\overline{c_{st}}$ if and only if the last $n-1$ rows of g are equal to the first $n-1$ rows of h. Hence, the set of $(SL(n), c_{st})$-bands over K can be identified with the set of $\infty \times n$ matrices

$$\mathrm{B} = (\mathrm{b_{ij}})_{\mathrm{i}\in\mathbb{Z}, 1\leq \mathrm{j} \leq \mathrm{n}}, \qquad (\mathrm{b_{ij}} \in \mathrm{K})$$

such that the sub-matrices

$$\mathrm{B(s)} := (\mathrm{b_{ij}})_{\mathrm{s}\leq \mathrm{i} \leq \mathrm{s+n-1}, 1\leq \mathrm{j}\leq \mathrm{n}} \qquad (\mathrm{s} \in \mathbb{Z})$$

have determinant equal to one. This identification assigns to a matrix B the band $(\mathrm{B(s)})_{\mathrm{s}\in\mathbb{Z}}$. For instance, the matrix

$$\mathrm{B} = \begin{pmatrix} \vdots & \vdots \\ 3 & 4 \\ 2 & 3 \\ 1 & 2 \\ 2 & 5 \\ 3 & 8 \\ \vdots & \vdots \end{pmatrix}, \qquad \left(\mathrm{B}(0) = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \right)$$

corresponds to the $(SL(2), c_{st})$-band whose components are:

$$g(-s) := \begin{pmatrix} s+2 & s+3 \\ s+1 & s+2 \end{pmatrix}, \ (s \geq 0), \qquad g(s) := \begin{pmatrix} s & 3s-1 \\ s+1 & 3s+2 \end{pmatrix}, \ (s > 0).$$

In this identification:

- The function $\Delta^{(s)}_{v\varpi_i, w\varpi_i}$ corresponds to the minor of B(s) taken along rows $v \cdot \{1, \ldots, i\}$ and columns $w \cdot \{1, \ldots, i\}$.
- The function $\theta^{(s)}_{i,k}$ sends a matrix B to the determinant of the maximal square submatrix of B taken along rows
$$([s, \ s+i-1] \sqcup [s+i+k, \ s+n+k-1]) \cap \mathbb{Z}.$$
- The action of $SL(n)$ on $B(SL(n), c_{st})$ is given by matrix multiplication.

The three cluster structures

To the Coxeter element $c = s_{i_1} \cdots s_{i_n}$ $(n = \mathrm{rk(G)})$ we associate an orientation Q of the Dynkin diagram of G by requiring that i_1 is a source of Q, i_2 is a source of $s_{i_1}(Q)$, and so on. Let KQ be the path algebra of Q and $\mathrm{D^b(KQ)}$ be the bounded derived category of $KQ-\mathrm{mod}$. We consider the following three quivers:

1. Λ : the product of a sink-source orientation of a Dynkin diagram of type A_∞ with a sink source orientation of the Dynkin diagram of G.
2. $\mathcal{G}_{\mathrm{DQ}}$: the Auslander-Reiten quiver of $\mathrm{D^b(KQ)}$ with additional arrows corresponding to AR-translation.

The quiver $\mathcal{G}_{\mathrm{DQ}}$ contains a copy of the Auslander-Reiten quiver $\mathcal{G}_Q$ of $KQ-\mathrm{mod}$. The vertices of this copy are coloured in red in the example on the right.

3. Γ_c : the quiver obtained from $\mathcal{G}_{\mathrm{DQ}}$ by replacing every vertex v of $\mathcal{G}_Q$ by an arrow $v_1 \longrightarrow v_2$ (between two new vertices v_1, v_2), so that the oriented 3-cycles of $\mathcal{G}_{\mathrm{DQ}}$ incident at v and modified by this procedure become oriented 4-cycles.

Definition: Let $\mathcal{A}$ be the cluster algebra of the quiver Λ, $\mathcal{B}$ be the **upper** cluster algebra of the quiver $\mathcal{G}_{\mathrm{DQ}}$, and $\mathcal{C}$ be the cluster algebra of the quiver Γ_c.

Theorem: There is an algebra isomorphism $K \otimes \mathcal{A} \longrightarrow R(G, c)^G$ sending the initial cluster variables to elements of the form

$$\theta^{(s)}_{i,k}, \qquad (s \in \mathbb{Z}, \ i \in I, \ k \in \mathbb{Z}_{>0}).$$

Theorem: There is an algebra isomorphism $K \otimes \mathcal{B} \longrightarrow R(G, c)^U$ sending the initial cluster variables to elements of the form

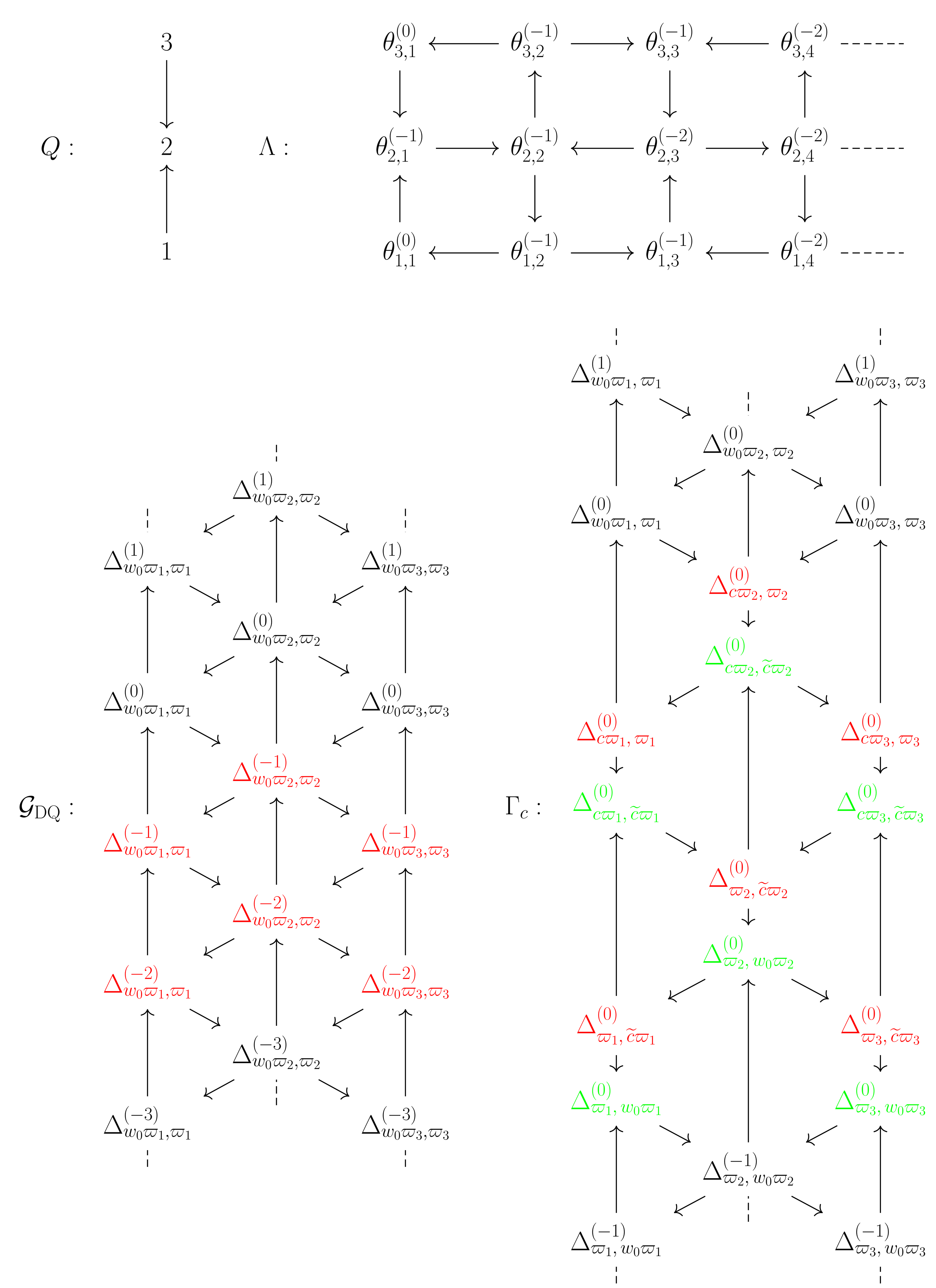
$$\Delta^{(s)}_{\varpi_i, \varpi_i}, \qquad (s \in \mathbb{Z}, \ i \in I).$$

Theorem: There is an algebra isomorphism $K \otimes \mathcal{C} \longrightarrow R(G, c)$ sending the initial cluster variables to elements of the form

$$\Delta^{(s)}_{c^k(\varpi_i), \widetilde{c}^\ell(\varpi_i)}, \qquad (s \in \mathbb{Z}, \ i \in I, k, \ell \in \mathbb{Z}),$$

An example in type A_3

Let $G = SL(4)$ and $c = s_1 s_3 s_2$, so that $\widetilde{c} = s_2 s_1 s_3$. The following picture describes some finite sections of the quivers Λ, $\mathcal{G}_{\mathrm{DQ}}$ and Γ_c. Vertices are labelled by the cluster variables of the corresponding initial seeds in $R(G, c)^G$, $R(G, c)^U$ and $R(G, c)$.



References

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