Silting objects, torsion classes, and cotorsion classes

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Context

Let Λ be a finite-dimensional algebra over an algebraically closed field k and $\mathcal{T} := K^b(\operatorname{proj}\Lambda)$.

An object $M \in \mathcal{T}$ is called a **silting object** if:

1. $\operatorname{Hom}(M, \Sigma^i M) = 0$ for all $i > 0$;
2. $\mathcal{T} = \operatorname{thick}(M)$.

It is called a d-**term silting object** if $M^i = 0$ for all $i \notin \{-(d-1), \cdots, -1, 0\}$.

Theorem 1: [AIR14][PZ23][Gar23]

We have the following diagram of morphisms of posets, where H^0 denotes the 0th cohomology functor:

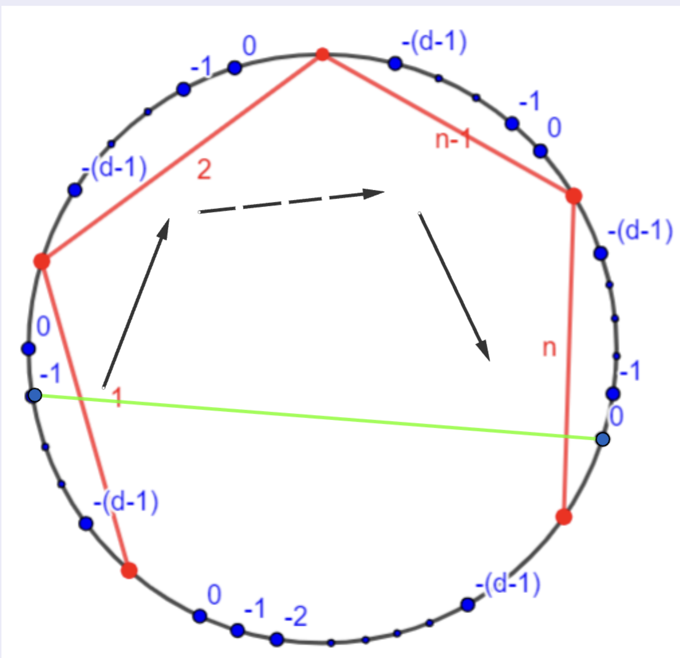
$$\begin{array}{ccc}
\{\textit{Basic 2-term silting objects in } \mathrm{K}^b(\operatorname{proj}\Lambda)\} & & \\
\updownarrow\wr & & \\
\{\textit{Complete cotorsion pairs in } \mathrm{K}^{[-1,0]}(\operatorname{proj}\Lambda)\} & \hookrightarrow & \{\textit{Cotorsion pairs in } \mathrm{K}^{[-1,0]}(\operatorname{proj}\Lambda)\} \\
\updownarrow\wr & & \wr\updownarrow \mathrm{H}^0 \\
\{\textit{Functorially finite torsion pairs in } \operatorname{mod}\Lambda\} & \hookrightarrow & \{\textit{Torsion pairs in } \operatorname{mod}\Lambda\}
\end{array}$$

Generalisations to d-term silting objects

Fix $d \geq 2$.

1. $\operatorname{mod}\Lambda \rightsquigarrow \mathcal{D}^{[-d+2,0]}(\operatorname{mod}\Lambda)$,
$$\mathcal{D}^{[-(d-2),0]}(\operatorname{mod}\Lambda) := \{X \in \mathcal{D}^b(\operatorname{mod}\Lambda) \mid \mathrm{H}^{>0}(X) = 0, \mathrm{H}^{<-d+2}(X) = 0\}$$
   - Extension-closed in $\mathcal{D}^b(\operatorname{mod}\Lambda) \implies$ Has the structure of an extriangulated category with negative extensions.
2. torsion pairs in $\operatorname{mod}\Lambda \rightsquigarrow$ torsion pairs in $\mathcal{D}^{[-d+2,0]}(\operatorname{mod}\Lambda)$
3. $\mathrm{H}^0 \rightsquigarrow \tau_{\geq -d+2}$, the canonical truncation functors.

A geometric model for silting objects in type A_n $(1 \to 2 \to \cdots \to n)$ (Inspired from [PPP22], [OPS18], [BC19])

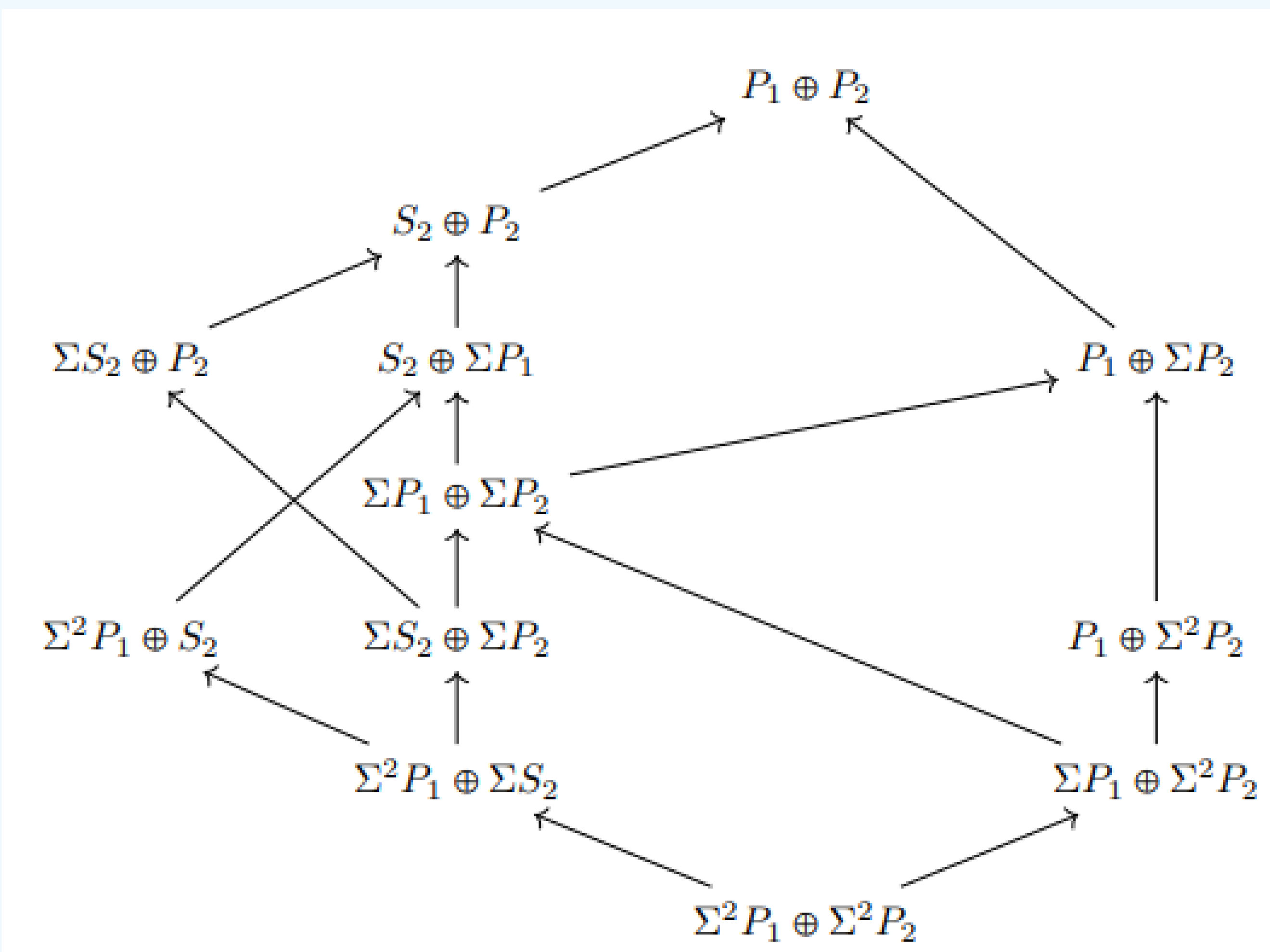


1. Mark $n+1$ red points on the boundary of a disc, and connect all pairs of adjacent points except one.
2. Mark blue points indexed from $-d+1$ to 0 in a clockwise direction between every pair of adjacent red points.
3. Special arcs connecting two blue points called **slaloms** are in bijection with indecomposable objects in $K^b(\operatorname{proj} kA_n)$: The green arc in the above figure is a slalom and corresponds to the complex $\cdots \to 0 \to P_1 \to P_n \to 0 \to \cdots$ concentrated in degrees $-1, 0$.

Main results: Type A_n

1. Two slaloms γ, γ' intersect in the interior of the disc if and only if there is a positive extension between the corresponding complexes in $K^b(\operatorname{proj}\Lambda)$.
2. Basic d-term silting objects are in bijection with collections of n slaloms that do not intersect in the interior of the disc in the above model.
3. The number of such collections can be calculated recursively using a cutting procedure, recovering the result of [STW20] that they are enumerated by the **Fuss-Catalan numbers**.

Mutation quiver of 3-term silting objects in kA_2



(Positive) Torsion pairs

Let $\mathcal{C}$ be an extriangulated category with a bivariant δ-functor $\mathbb{E}^i(-,-)$ for $i \in \mathbb{Z}$. Let $\mathcal{T}, \mathcal{F} \subseteq \mathcal{C}$ be two full subcategories. The pair $(\mathcal{T}, \mathcal{F})$ is called a *torsion pair* if

1. $\mathcal{T} = {}^{\perp}\mathcal{F}$,
2. $\mathcal{F} = \mathcal{T}^{\perp}$.

A torsion pair is called *positive* if, additionally,

3. $\mathbb{E}^i(\mathcal{T}, \mathcal{T}^{\perp}) = 0$ for all $i \leq 0$.

(Hereditary) Cotorsion pairs [NP19]

Let $\mathcal{C}$ be an extriangulated category and $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{C}$ be full subcategories. Then the pair $(\mathcal{X}, \mathcal{Y})$ is called a *cotorsion pair* if

1. $\mathcal{X} = {}^{\perp_1}Y$,
2. $\mathcal{Y} = \mathcal{X}^{\perp_1}$.

A cotorsion pair is called *hereditary* if, additionally,

3. $\mathbb{E}^i(\mathcal{X}, \mathcal{Y}) = 0$ for all $i \geq 1$.

Theorem 2: [KY14]

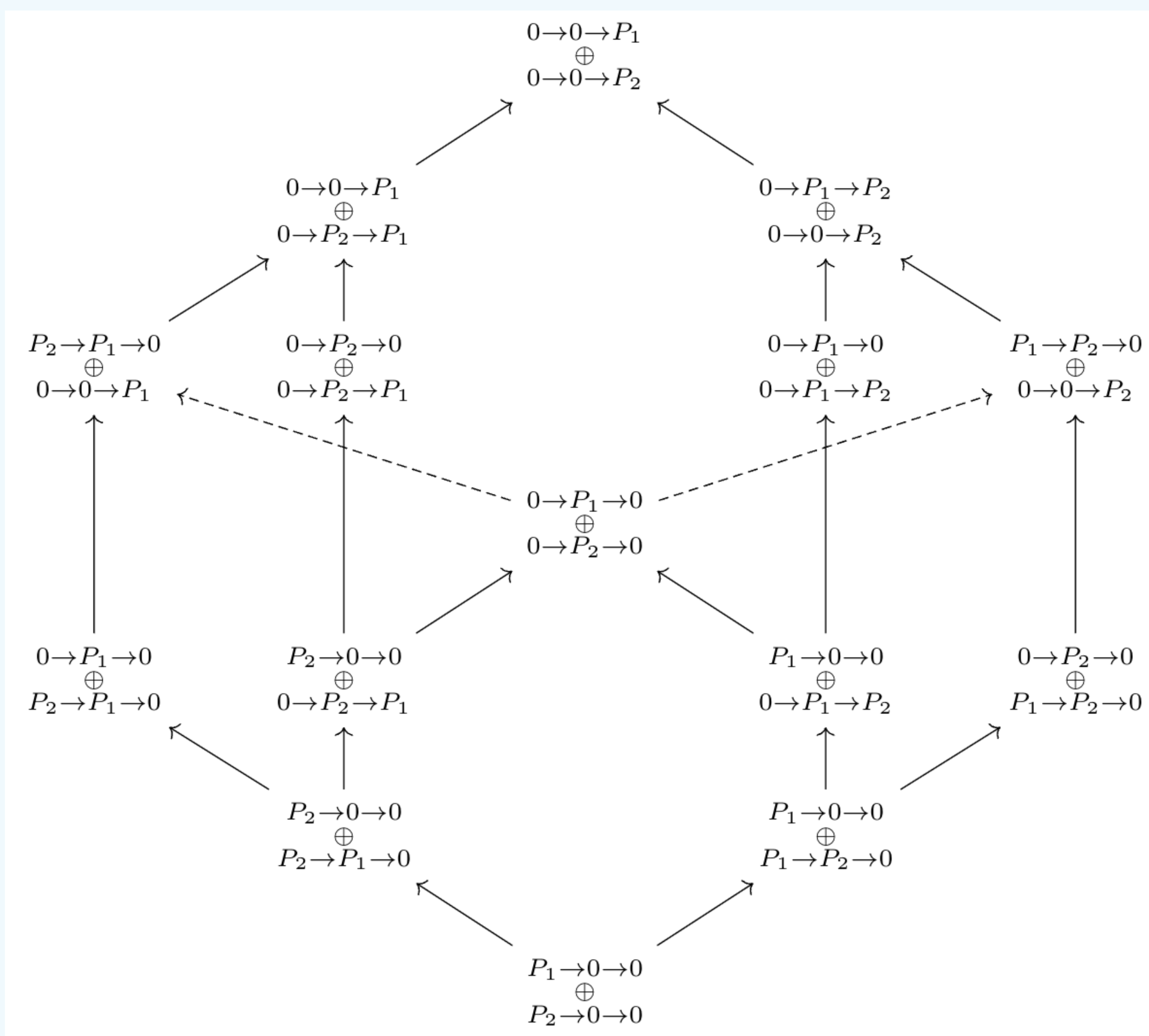
The following posets are isomorphic.

$$\begin{array}{c}
\{\textit{equivalence classes of silting objects in } K^b(\operatorname{proj}\Lambda)\} \\
\updownarrow \\
\{\textit{bounded } t\textit{-structures in } \mathcal{D}^b(\operatorname{mod}\Lambda) \textit{ with length heart}\} \\
\updownarrow \\
\{\textit{bounded co-}t\textit{-structures in } \mathrm{K}^b(\operatorname{proj}\Lambda)\}
\end{array}$$

Main results: General case

1. The functor $\tau_{\geq -d+2}$ induces the following equivalence of additive categories.
$$\tau_{\geq -d+2} : \frac{\mathrm{K}^{[-d+1,0]}(\operatorname{proj}\Lambda)}{\operatorname{proj}\Lambda[d-1]} \xrightarrow{\sim} \mathcal{D}^{[-d+2,0]}(\operatorname{mod}\Lambda)$$
2. We have the following diagram of morphisms of posets:
$$\begin{array}{ccc}
d\text{-}\operatorname{silt}\Lambda & & \\
\updownarrow\wr & & \\
(\text{comp, her})\text{-cotors}\,\mathrm{K}^{[-d+1,0]}(\operatorname{proj}\Lambda) & \xleftrightarrow{\sim} & (\text{ff, pos})\text{-tors}\,\mathcal{D}^{[-d+2,0]}(\operatorname{mod}\Lambda) \\
\downarrow & & \downarrow \\
\text{her-cotors}\,\mathrm{K}^{[-d+1,0]} & \xleftrightarrow{\sim} & \text{pos-tors}\,\mathcal{D}^{[-d+2,0]}(\operatorname{mod}\Lambda) \\
\downarrow & & \downarrow \\
\text{cotors}\,\mathrm{K}^{[-d+1,0]}(\operatorname{proj}\Lambda) & \xleftrightarrow{\sim} & \text{tors}\,\mathcal{D}^{[-d+2,0]}(\operatorname{mod}\Lambda)
\end{array}$$
where her: hereditary, comp: complete, ff: functorially finite, and pos: positive.
3. The posets of (positive) torsion pairs in $\mathcal{D}^{[-d+2,0]}(\operatorname{mod}\Lambda)$ and (hereditary) cotorsion pairs in $\mathrm{K}^{[-d+1,0]}(\operatorname{proj}\Lambda)$ are, in fact, lattices.

A non-hereditary example



$$3\text{-silt}\, k\Big(\begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] & 2 \arrow[l, bend left, "\beta"] \end{tikzcd} \Big)/\langle \alpha\beta, \beta\alpha \rangle$$

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